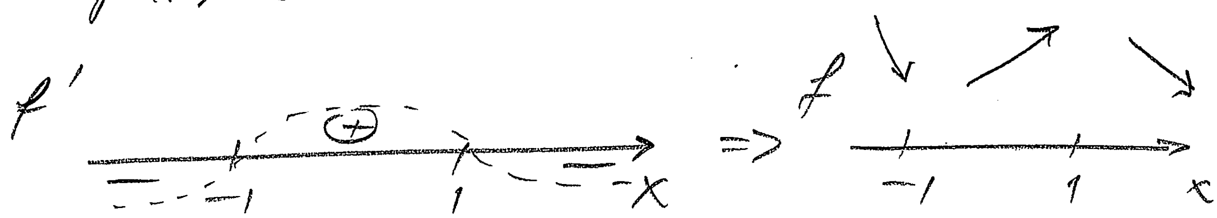
$\boxed{\text{Problem \#7 (4.2)}}$

$$f(x) = \frac{4x}{x^2+1}$$

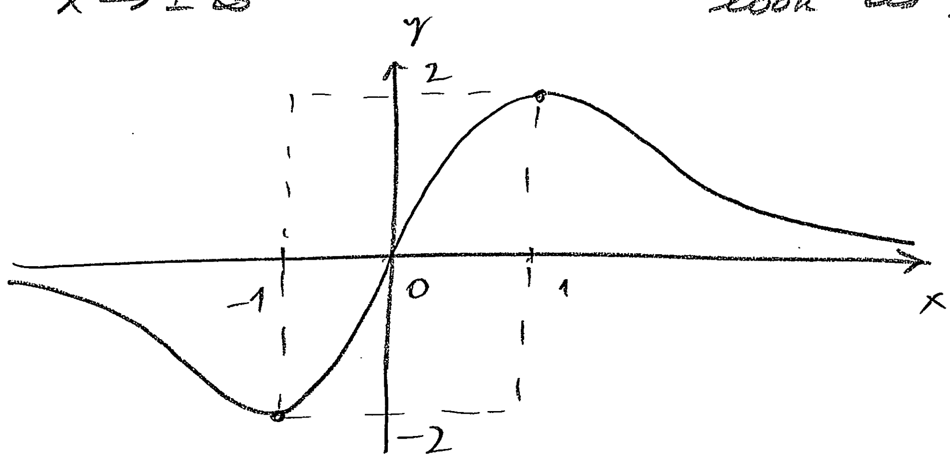
$$f'(x) = \frac{4(x^2+1) - 4x \cdot 2x}{(x^2+1)^2}$$

$$= \frac{-4x^2+4}{(x^2+1)^2} = -4\,\frac{x^2-1}{(x^2+1)^2}.$$

$$f'(x) = 0 \iff x = \pm 1$$



Since $f(-1) = -2$, $f(0) = 0$, $f(1) = 2$, $\lim_{x \to \pm\infty} f(x) = 0$, the graph must look as follows:



$I_1 = (-\infty, -1)$ (f decr.) ; $I_2 = (-1, 1)$ (f incr.) ; $I_3 = (1, \infty)$ (f decr.)

$J_1 = (-2, 0)$; $J_2 = (-2, 2)$; $J_3 = (0, 2)$

Expressions for g_1, g_2, g_3:

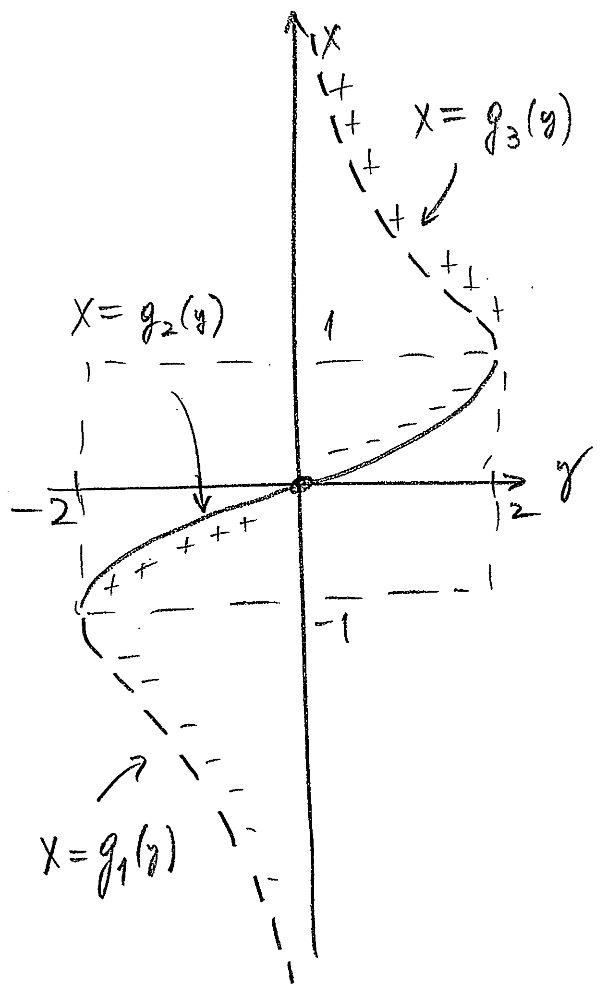
Solve

$$\frac{4x}{x^2+1} = y$$

$$\frac{4x}{y} = x^2+1 \Rightarrow x^2 - \frac{4x}{y} + 1 = 0$$

$$\Rightarrow x = \frac{2}{y} \pm \sqrt{\frac{4}{y^2} - 1} = \frac{2 \pm \sqrt{4-y^2}}{y}$$

x is defined for $|y| < 2$



$$g_1(y) = \frac{2 - \sqrt{4-y^2}}{y}, \quad y \in (-2, 0)$$

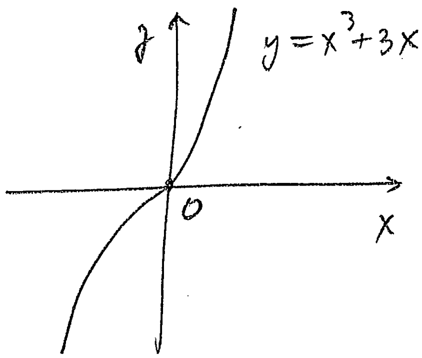
$$g_2(y) = \begin{cases} \dfrac{2 + \sqrt{4-y^2}}{y}, & y \in (-2, 0) \\ 0, & y = 0 \\ \dfrac{2 - \sqrt{4-y^2}}{y}, & y \in (0, 2) \end{cases}$$

$$g_3(y) = \frac{2 + \sqrt{4-y^2}}{y}, \quad y \in (0, 2).$$

Problem #9 (4.2.)

$f(x) = x^3 + 3x$

$f'(x) = 3x^2 + 3 > 0$



$\Rightarrow$ $f(x)$ is increasing on $(-\infty, \infty)$

$\Rightarrow$ The inverse g is defined on $(-\infty, \infty)$. (the range of f).

Expression for g:

Solve $x^3 + 3x = y$ "depressed cubic".

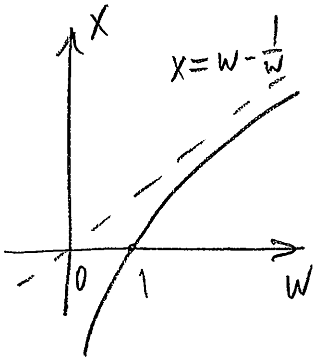
Substitute $x = w - \frac{1}{w}$ ("Tartaglia's substitution")

$$x^3 + 3x = w^3 - 3w^2\frac{1}{w} + 3w\frac{1}{w^2} - \frac{1}{w^3} + 3w - \frac{3}{w}$$

$$= w^3 - \frac{1}{w^3} = y$$

$$\Rightarrow w^6 - yw^3 - 1 = 0$$

$$w^3 = \frac{y}{2} \pm \sqrt{\frac{y^2}{4} + 1} = \frac{y \pm \sqrt{y^2+4}}{2}$$



The "+" sign is relevant for this problem: then $w > 0$ and $y > 0$ corresponds to $w > 1 \Rightarrow x > 0$ and $y < 0 \Rightarrow w < 1 \Rightarrow x < 0$. (See figure.)

Also $\frac{y + \sqrt{y^2+4}}{2} = \frac{(y + \sqrt{y^2+4})}{2}\frac{(y - \sqrt{y^2+4})}{(y - \sqrt{y^2+4})}$

$$= \frac{y^2 - y^2 - 4}{2\left(y - \sqrt{y^2+4}\right)} = \frac{2}{\sqrt{y^2+4} - y}$$

$$\Rightarrow \quad x = w - \frac{1}{w} = \left(\frac{y + \sqrt{y^2+4}}{2}\right)^{1/3} - \left(\frac{\sqrt{y^2+4} - y}{2}\right)^{1/3}.$$

Problem #14 (4.2)

$$f(x) = \frac{4x}{x^2+1}, \quad I = (1, \infty)$$

(typo in the textbook, f is not monotone on $(\frac{1}{2}, \infty)$)

$$f'(x) = -4 \frac{x^2-1}{(x^2+1)^2}$$

$$g_3(y) = \frac{2}{y} + \sqrt{\frac{4}{y^2} - 1} = \frac{2+\sqrt{4-y^2}}{y}$$

$$g_3'(y) = \left(\frac{2}{y} + \sqrt{\frac{4}{y^2}-1}\right)' = -2y^{-2} - \frac{4y^{-3}}{\sqrt{4y^{-2}-1}}$$

$$= -\frac{2}{y^2} - \frac{4}{y^2\sqrt{4-y^2}} = -\frac{2\sqrt{4-y^2}+4}{y^2\sqrt{4-y^2}}.$$

Check the derivative of the Inverse Function formula:

$$g_3'(y) = \frac{1}{f'(g_3(y))} = -\frac{1}{4}\frac{(x^2+1)^2}{x^2-1}$$

$$x = g_3(y) = \frac{2+\sqrt{4-y^2}}{y}$$

$$x^2 = \left(\frac{2+\sqrt{4-y^2}}{y}\right)^2 = \frac{4+4-y^2+4\sqrt{4-y^2}}{y^2}$$

$$x^2+1 = \frac{8+4\sqrt{4-y^2}}{y^2} = \frac{4\left(2+\sqrt{4-y^2}\right)}{y^2}$$

$$x^2-1 = \frac{8-2y^2+4\sqrt{4-y^2}}{y^2} = \frac{2(4-y^2)+4\sqrt{4-y^2}}{y^2}$$

$$= \frac{2\sqrt{4-y^2}}{y^2}\left(2+\sqrt{4-y^2}\right).$$

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$$g_3'(y) = -\frac{1}{4}\,\frac{(x^2+1)^2}{x^2-1} = -\frac{1}{4}\,\frac{16}{y^4}\,(2+\sqrt{4-y^2})^2\,\frac{y^2}{2\sqrt{4-y^2}\,(2+\sqrt{4-y^2})}$$

$$= -2\,\frac{2+4\sqrt{4-y^2}}{y^2\sqrt{4-y^2}}. \quad \checkmark$$

$\boxed{\text{Problem \#16 (4.2)}}$

$$f(x) = x^3 + 3x, \quad x \in (-\infty, \infty).$$

$$f'(x) = 3x^2 + 3$$

Inverse function:

$$g(y) = \left(\frac{\sqrt{y^2+4}+y}{2}\right)^{1/3} - \left(\frac{\sqrt{y^2+4}-y}{2}\right)^{1/3}.$$

$$g'(y) = \frac{1}{3}\left(\frac{\sqrt{y^2+4}+y}{2}\right)^{-2/3}\left(\frac{y}{2\sqrt{y^2+4}} + \frac{1}{2}\right)$$

$$- \frac{1}{3}\left(\frac{\sqrt{y^2+4}-y}{2}\right)^{-2/3}\left(\frac{y}{2\sqrt{y^2+4}} - \frac{1}{2}\right)$$

$$= \frac{1}{3}\left(\frac{\sqrt{y^2+4}+y}{2}\right)^{-2/3}\frac{(\sqrt{y^2+4}+y)}{2\sqrt{y^2+4}}$$

$$+ \frac{1}{3}\left(\frac{\sqrt{y^2+4}-y}{2}\right)^{-2/3}\frac{(\sqrt{y^2+4}-y)}{2\sqrt{y^2+4}}$$

$$= \frac{1}{3}\frac{1}{\sqrt{y^2+4}}\left(\left(\frac{\sqrt{y^2+4}+y}{2}\right)^{1/3} + \left(\frac{\sqrt{y^2+4}-y}{2}\right)^{1/3}\right)$$

$$= \frac{1}{3}\frac{1}{\sqrt{y^2+4}}\frac{\frac{\sqrt{y^2+4}+y}{2} + \frac{\sqrt{y^2+4}-y}{2}}{\left(\frac{\sqrt{y^2+4}+y}{2}\right)^{2/3} - \underbrace{\left(\frac{\sqrt{y^2+4}+y}{2}\right)^{1/3}\left(\frac{\sqrt{y^2+4}-y}{2}\right)^{1/3}}_{=1} + \left(\frac{\sqrt{y^2+4}-y}{2}\right)^{2/3}}$$

$$= \frac{1}{3}\frac{1}{\left(\frac{\sqrt{y^2+4}+y}{2}\right)^{2/3} - 1 + \left(\frac{\sqrt{y^2+4}-y}{2}\right)^{2/3}}.$$

Check the derivative of the inverse function formula:

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{3x^2+3} = \frac{1}{3}\frac{1}{x^2+1}$$

$$= \frac{1}{3} \, \frac{1}{\left(\left(\frac{\sqrt{y^2+4}+y}{2} \right)^{1/3} - \left(\frac{\sqrt{y^2+4}-y}{2} \right)^{1/3} \right)^2 + 1}$$

$$= \frac{1}{3} \, \frac{1}{\left(\frac{\sqrt{y^2+4}+y}{2} \right)^{2/3} - 2 \underbrace{\left(\frac{\sqrt{y^2+4}+y}{2} \right)^{1/3} \left(\frac{\sqrt{y^2+4}-y}{2} \right)^{1/3}}_{=1} + \left(\frac{\sqrt{y^2+4}-y}{2} \right)^{2/3} + 1}$$

$$= \frac{1}{3} \, \frac{1}{\left(\frac{\sqrt{y^2+4}+y}{2} \right)^{2/3} - 1 + \left(\frac{\sqrt{y^2+4}-y}{2} \right)^{2/3}} .$$