This exam includes 8 questions (the last one is a bonus). Please check that your copy has 9 pages. The duration of the exam is 1 hour 15 minutes.

Your scores: (do not enter answers here)

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Important: The exam is closed books/notes. Graphing calculators are permitted. Show all your work.

1. (5 points) Find a second order linear differential equation such that \( u_1(t) = \cos(5 \ln t) \), \( u_2(t) = \sin(5 \ln t) \) are two linearly independent solutions.

Cauchy-Euler equation;
char. equation has roots \( m_{1,2} = \pm 5i \):

\[
m^2 + 25 = 0
\]
\[
m(m-1) + \frac{1}{t^2} + 25 = 0
\]
\[
t^2 u'' + t u' + 25 u = 0
\]

(Check: \( u'_1(t) = -\frac{5}{t} \sin(5 \ln t) \)
\[
u''_1(t) = \frac{5}{t^2} \sin(5 \ln t) - \frac{25}{t^2} \cos(5 \ln t)
\]
\[
t^2 u''_1 + t u'_1 = -25 \cos(5 \ln t) = -25 u_1
\]

Likewise for \( u_2(t) \).)
2. (10 points)

(a) Formulate the conditions under which all solutions of the equation \( a(t)u'' + b(t)u' + c(t)u = 0 \) on the interval \((t_1, t_2)\) are given by

\[ u(t) = c_1 u_1(t) + c_2 u_2(t), \]

where \( u_1(t), u_2(t) \) are two arbitrary linearly independent solutions.

\[ \frac{b(t)}{a(t)} \quad \text{and} \quad \frac{c(t)}{a(t)} \quad \text{continuous on} \quad (t_1, t_2). \]

(b) Consider the functions \( u_1(t) = t^3, u_2(t) = |t|^3, \) and \( u_3(t) = t \). Show that they are linearly independent on \((-1, 1)\).

\[ \begin{align*}
\text{Assume} & \quad \zeta_1 t^3 + \zeta_2 |t|^3 + \zeta_3 t = 0, \quad t \in (-1, 1). \\
\text{If} & \quad t > 0, \quad (\zeta_1 + \zeta_2) t^3 + \zeta_3 t = 0 \\
& \quad \implies \zeta_1 + \zeta_2 = 0, \quad \zeta_3 = 0. \\
\text{If} & \quad t < 0, \quad (\zeta_1 - \zeta_2) t^3 + \zeta_3 t = 0 \\
& \quad \implies \zeta_1 - \zeta_2 = 0, \quad \zeta_3 = 0. \\
\zeta_1 + \zeta_2 = 0 & \quad \implies \zeta_1 = \zeta_2 = 0. \\
\zeta_1 - \zeta_2 = 0 & \quad \implies \zeta_1 = \zeta_2 = 0. 
\end{align*} \]

Continued...
(c) Find a differential equation of the form \( a(t)u'' + b(t)u' + c(t)u = 0 \) such that \( u_1(t) = t^3 \), \( u_2(t) = |t|^3 \), and \( u_3(t) = t \) are solutions for \( t \in (-1, 1) \). [Hint: A Cauchy-Euler equation would do.]

\[
t^m, \quad m = 1, 3 \quad \text{solutions}; \\
(m-1)(m-3) = m(m-1) - 3m + 3 \\
t^2u'' - 3tu' + 3u = 0
\]

\( u_2(t) = \begin{cases} 
  t^3 & t > 0 \\
  -t^3 & t < 0 
\end{cases} \Rightarrow \text{solution on } (0,1) \text{ and on } (-1,0) \).

Since \( u_2(t) \) is continuously differentiable at 0, it is a solution on \((-1, 1)\).

(d) Which of the conditions of part (a) are not satisfied by the differential equation in part (c)?

\[
\frac{b(t)}{a(t)} = -\frac{3}{t} \quad \text{not continuous at } t = 0. \\
\frac{c(t)}{a(t)} = \frac{3}{t^2}
\]
3. (8 points) Solve the initial-value problem

\[ u'' + 2u' + u = 2e^{-t}, \quad u(0) = 1, \quad u'(0) = 2. \]

**homogeneous problem:**

\[ u'' + 2u' + u = 0 \]

\[ \lambda^2 + 2\lambda + 1 = 0 \]

\[ (\lambda + 1)^2 = 0 \quad \Rightarrow \quad \lambda = -1. \quad (\text{multiplicity 2}) \]

\[ u_h(t) = c_1 e^{-t} + c_2 t e^{-t}. \]

**particular solution:**

\[ u_p(t) = at^2 e^{-t} \]

\( t^2 \) is the lowest power such that \( u_p(t) \) is not part of homogeneous solution.

\[ u_p' = -at^2 e^{-t} + 2at e^{-t} \]

\[ u_p'' = at^2 e^{-t} - 4at e^{-t} + 2ae^{-t} \]

\[ u_p'' + 2u_p' + u_p = 2ae^{-t} = 2e^{-t} \]

\[ \Rightarrow a = 1 \]

**Answer:**

\[ u(t) = u_p(t) + u_h(t) \]

\[ = t^2 e^{-t} + c_1 e^{-t} + c_2 t e^{-t} \]

Continued...
4. (10 points) Find all values of \( \lambda \) for which the boundary-value problem

\[
-u'' = \lambda u, \quad 0 < x < \pi
\]
\[
u'(0) = 0, \quad u(\pi) = 0
\]

has nontrivial solutions, and determine those solutions.

**Equation** \(-u'' = \lambda u\) **has solutions**

\[
u(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x) \quad (\lambda > 0)
\]
\[
u(x) = C_1 + C_2 x \quad (\lambda = 0)
\]
\[
u(x) = C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x} \quad (\lambda < 0)
\]

In the last two cases the only solution that satisfies boundary conditions is zero:

\[
\lambda = 0:
\]
\[
u'(x) = C_2 \quad \nu'(0) = C_2 = 0
\]
\[
u(x) = C_1 \quad \nu(\pi) = C_1 = 0
\]

\[
\lambda < 0:
\]
\[
u'(x) = C_1 \sqrt{-\lambda} e^{\sqrt{-\lambda} x} - C_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda} x}
\]
\[
u'(0) = C_1 \sqrt{-\lambda} - C_2 \sqrt{-\lambda} = 0
\]
\[
u(x) = C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x}
\]
\[
\begin{vmatrix}
 e^{\sqrt{-\lambda} \pi} & e^{-\sqrt{-\lambda} \pi} \\
 e^{-\sqrt{-\lambda} \pi} & e^{\sqrt{-\lambda} \pi}
\end{vmatrix} = 0
\]

\[
\Rightarrow C_1 = C_2 = 0
\]

**If** \( \lambda > 0 \),

\[
u(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)
\]
\[
u'(x) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)
\]
\[
u'(0) = C_2 \sqrt{\lambda} = 0 \Rightarrow C_2 = 0
\]
\[
u(\pi) = C_1 \cos(\sqrt{\lambda} \pi) = 0
\]
\[
\sqrt{\lambda} \pi = \frac{\pi}{2} + n\pi \Rightarrow \lambda = \left(\frac{1}{2} + n\right)^2
\]
\[n = 0, 1, 2, \ldots
\]

\[
u(x) = C_1 \cos\left((\frac{1}{2} + n) x\right), \quad n = 0, 1, 2, \ldots
\]

Continued...
5. (10 points) Find the general solution of \( xy'' + 2y' - xy = 0 \) \((x > 0)\) given that \( y_1(x) = \frac{e^x}{x} \) is a particular solution.

\[
y(x) = y_1(x) v(x)
\]

\[
y' = y_1' v + y_1 v'
\]

\[
y'' = y_1'' v + 2y_1' v' + y_1 v''
\]

\[
xy'' + 2y' - xy = x(2y_1' v + y_1 v'') + 2y_1 v' = 0
\]

\[
v' = w \Rightarrow w' + \left(\frac{2}{x} + 2\frac{y_1'}{y_1}\right)w = 0
\]

\[
\int \frac{w'dx}{w} = - \int \left(\frac{2}{x} + 2\frac{y_1'}{y_1}\right)dx
\]

\[
\ln w = -2 \ln x - 2 \ln y_1
\]

\[
w = \frac{1}{y_1^2} e^{-2\ln x} = \frac{1}{y_1^2 x^2} = \frac{1}{e^{2x}}
\]

\[
v = \int w \, dx = -\frac{1}{x} e^{-2x}
\]

\[
y_2(x) = -\frac{1}{x} \frac{e^{-x}}{x}
\]

\[
y(x) = c_1 \frac{e^x}{x} + c_2 \frac{e^{-x}}{x}
\]

- general solution.

Continued...
6. (10 points) Use the power series method to find the first three terms of two linearly independent solutions to $u'' + tu' + tu = 0$ valid near $t = 0$.

$$u(t) = \sum_{n=0}^{\infty} a_n t^n, \quad t u(t) = \sum_{n=1}^{\infty} a_{n-1} t^n$$

$$u'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1},$$

$$t u'(t) = \sum_{n=1}^{\infty} n a_n t^n$$

$$u''(t) = \sum_{n=0}^{\infty} \frac{(n+2)(n+1) a_{n+2} + n a_n}{n} t^n$$

$$u'' + tu' + tu = 2a_2 + \sum_{n=1}^{\infty} \left[ \frac{(n+2)(n+1) a_{n+2} + n a_n}{n} t^n + a_{n-1} \right] t^n$$

$$a_2 = 0,$$

$n=1$: $3.2 \cdot a_3 + a_1 + a_0 = 0 \Rightarrow a_3 = -\frac{1}{6} a_0 - \frac{1}{6} a_1$

$n=2$: $4.3 \cdot a_4 + 2a_2 + a_1 = 0 \Rightarrow a_4 = -\frac{1}{12} a_1$

$n=3$: $5.4 \cdot a_5 + 3a_3 + a_2 = 0 \Rightarrow a_5 = \frac{1}{40} a_0 + \frac{1}{40} a_1$

$$u(x) = a_0 \left( 1 - \frac{1}{6} x^3 + \frac{1}{40} x^5 + \ldots \right)$$

$$+ a_1 \left( x - \frac{1}{6} x^3 - \frac{1}{12} x^5 + \ldots \right)$$

Continued...
7. (10 points) An LC circuit contains a capacitor with \( C = 0.25 \) and an inductor with \( L = 1 \) connected in series with a 10V battery. Find the current \( I(t) = q'(t) \) for \( t > 0 \) if at time \( t = 0 \) both the current and the charge \( q(t) \) are zero. Sketch the graph of the function \( I(t) \). [Reminder: the voltage drop on a capacitor is \( V_C = \frac{q}{C} \) and the voltage drop on an inductor is \( V_L = L I' \).]

\[
E = 10V
\]

\[
C = 0.25
\]

\[
L = 1
\]

\[
I(t) = q'(t)
\]

\[
Lq'' + \frac{1}{C} q = E
\]

\[
q'' + 4q = 10
\]

**Particular solution:** \( q_p = \frac{10}{4} = 2.5 \)

**Homogeneous solution:** \( q_h(t) = C_1 \cos(2t) + C_2 \sin(2t) \)

\[
q(t) = q_p + q_h = \frac{5}{2} + C_1 \cos(2t) + C_2 \sin(2t)
\]

\[
q(0) = \frac{5}{2} + C_1 \Rightarrow C_1 = -\frac{5}{2}
\]

\[
q'(t) = 5 \sin(2t) + 2 \cdot C_2 \cos(2t)
\]

\[
q'(0) = 2C_2 = 0 \Rightarrow C_2 = 0.
\]

\[
q(t) = \frac{5}{2} (1 - \cos(2t)) \quad I(t) = q'(t)
\]

\[
I(t) = 5 \sin(2t)
\]

Continued...
8. (bonus: 10 points) Solve the Cauchy-Euler equation \( t^2 u'' - 3tu' + 3u = \frac{1}{t^2} \).

\[
\begin{align*}
m(m-1) - 3m + 3 &= 0 \\
m^2 - 4m + 3 &= 0 \\
(m - 1)(m - 3) &= 0 \\
m &= 1, 3 \\
\end{align*}
\]

Homogeneous solution: \( u_h = C_1 t + C_2 t^3 \).

Variation of parameters: \( u_p = C_1(t) t + C_2(t) t^3 \)

\[
\begin{align*}
C_1' t + C_2' t^3 &= 0 \\
C_1' t^2 + C_2' (t^3)' &= \frac{1}{t^4} \\
0 + 3t^2 C_2' &= \frac{1}{t^4} \\
C_2' &= \frac{1}{2t^6} \\
C_2 &= -\frac{1}{10t^4} \\
\end{align*}
\]

\[
\begin{align*}
u_p &= \frac{1}{6t^3} t - \frac{1}{10t^4} t^3 = (\frac{1}{6} - \frac{1}{10}) \frac{1}{t^2} \\
\text{Alternatively, undetermined coefficients:} \\
u_p &= \frac{C}{t^2} \\
u_p' &= -\frac{2C}{t^3}, \quad u_p'' = \frac{6C}{t^4} \\
t^2 u_p'' - 3tu_p' + 3u_p &= \frac{15C}{t^2} = \frac{1}{t^2} \implies C = \frac{1}{15} \\
\text{Answer: } u(t) = \frac{1}{15t^2} + C_1 t + C_2 t^3.
\end{align*}
\]