Homework Assignment 1

Due Tue. Feb. 2, 2010, in class.

1. Verify that the initial-value problems 

\[ u' = 1 - u^2, \quad u(0) = 0, \quad \text{and} \quad u' = 1 + u^2, \quad u(0) = 0 \]

have solutions 

\[ u(t) = e^{2t} - 1 \quad \text{and} \quad u(t) = \tan t \], respectively. What are the maximal intervals on which each of these solutions are defined and continuously differentiable?

2. Show that 

\[ u(t) = \ln(t + C) \]

is a one-parameter family of solutions of the DE \( u' = e^{-u} \) where \( C \) is an arbitrary constant. Plot several members of this family. Find and plot a particular solution that satisfies the initial condition \( u(0) = 0 \).

3. Find a solution 

\[ u = u(t) \]

of \( u' + 2u = t^2 + 4t + 7 \) in the form of a quadratic function of \( t \).

4. Plot the one-parameter family of curves 

\[ u(t) = (t^2 + C)e^t \]

and find a differential equation whose solution is this family. [Hint: the solution curves look different for \( C > 1 \), \( C = 1 \) and \( C < 1 \). Show all three types of solutions on your graph.]

5. Classify the first-order equations as linear or nonlinear, autonomous or nonautonomous.

(a) \( u' = 2t^3u - 6 \).

(b) \( (\cos t)u' - 2u \sin u = 0 \).

(c) \( u' = 1 - u^2 \).

(d) \( 7u' - 3u = 0 \).

6. Consider the linear differential equation \( u' = p(t)u + q(t) \). Is it true that the sum of two solutions is again a solution? Is a constant times a solution again a solution? Answer these same questions if \( q(t) = 0 \). Show that if \( u_1 \) is a solution to \( u' = p(t)u \) and \( u_2 \) is a solution to \( u' = p(t)u + q(t) \), then \( u_1 + u_2 \) is a solution to \( u' = p(t)u + q(t) \).

7. By hand, sketch the direction field for the DE \( u' = u(2-u) \) in the window \(-2 \leq t \leq 2, 0 \leq u \leq 3\) at half-integer points \(((0, 0), (0, \pm \frac{1}{2}), (\pm 1, 0), \text{etc.})\). What is the value of the slope along the lines \( u = 0 \) and \( u = 2 \)? Show that \( u(t) = 0 \) and \( u(t) = 2 \) are constant solutions to the DE. On your slope field plot, draw in several solution curves.

8. For the DE in the previous problem show that all solutions \( u(t) \) are convex \((u'' > 0)\) if \( u > 2 \) or \( 0 < u < 1 \), and they are concave \((u'' < 0)\) if \( 1 < u < 2 \). Check your graph from the previous problem to make sure the solution curves satisfy these conditions. Hint: differentiate the DE to find the equation satisfied by the second derivative.
9. For the DE \( u' = u + t \) sketch the lines in the \((t, u)\)-plane where the slope of the solution has a constant value (try values \(-3, -2, -1, 0, 1, 2\)...). Using the obtained information draw several approximate solution curves. (Lines and curves in the \((t, u)\)-plane where the slope of the solution is constant are called isoclines.)

10. Using antiderivatives, find the general solution to the pure time equation \( u = t \cos(t^2) \), and then find the particular solution satisfying the initial condition \( u(0) = 1 \). Graph the particular solution on the interval \([5, 5]\).

11. Solve the initial value problem \( u' = \frac{t+1}{\sqrt{t}} \), \( u(1) = 4 \).

12. Consider a damped spring-mass system whose position \( x(t) \) is governed by the equation \( mx'' = -cx' - kx \) (\( c \) and \( k \) are positive constants). Show that this equation can have a "decaying-oscillation" solution of the form \( x(t) = e^{-\lambda t} \cos \omega t \). (Hint: By substituting into the differential equations, show that the decay constant \( \lambda \) and frequency \( \omega \) can be determined in terms of the given parameters \( m, c, \) and \( k \).)