Wed, April 20, 2011 **MATH 350, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

CSUN ID No. :

This test includes 6 questions (52 points in total) in the main part, and one bonus question, worth an extra 6 points. Please check that your copy of the test has 7 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (8 points) Find the supremum and infimum of the set

$$S = \left\{ \sum_{j=1}^{n} 1/2^j \, : \, n \in \mathbb{N} \right\}.$$

Does the set S have a maximum and a minimum?

Let $x_n = \sum_{j=1}^{n} \frac{1}{2^j} = 1 - \frac{1}{2^n}$ (by the formula for the geometric progression.)

Then $\frac{1}{2} \le x_n \le 1$;

$(x_1 = \frac{1}{2})$

$A = \frac{1}{2}$ is the minimum of $\{x_n\}$ ($\Rightarrow$ infimum)

$B = 1$ is the supremum, without being a maximum

(For $\forall \varepsilon > 0$ $\exists n$: $\frac{1}{2^n} < \varepsilon \Rightarrow x_n = 1 - \frac{1}{2^n} > 1 - \varepsilon$

$\Rightarrow 1 - \varepsilon$ is not an upper bound

$\Rightarrow$ 1 is the least upper bound.)

2. (8 points) Answer <u>one of the following</u>:

   (a) Prove (based on the completeness axiom or other equivalent principle) that every nondecreasing bounded above sequence of real numbers is convergent.

   (b) Prove that a sequence of real numbers that has the Cauchy property is convergent.

(a) Completeness axiom: $\forall A, B \subset \mathbb{R}$, $\neq \emptyset$: $\forall x \in A\ \forall y \in B\ x \le y$

$\exists c \in \mathbb{R} : \forall x \in A\ \forall y \in B\ x \le c \le y.$

Let x_n be nondecreasing, bounded above.

Let $A = \{x_n, n \in \mathbb{N}\}$ and $B = \{y \in \mathbb{R} : \forall n \in \mathbb{N}\ x_n \le y\}$ (the set of upper bounds.)

Then A, B satisfy the conditions of the completeness axiom

$\Rightarrow \exists c : \forall n \in \mathbb{N}, \forall y \in B \quad x_n \le c \le y.$

$\forall \varepsilon > 0 \quad c - \varepsilon$ is not an upper bound of A

$\Rightarrow \exists n_\varepsilon : \quad x_{n_\varepsilon} > c - \varepsilon.$

Since x_n is nondecreasing, $\forall n > n_\varepsilon$

$$c - \varepsilon < x_{n_\varepsilon} \le x_n \le c$$

$$\Rightarrow \quad -\varepsilon < x_n - c \le 0$$

$$\Rightarrow \quad |x_n - c| < \varepsilon$$

So the sequence x_n has the value c as the limit.

(b) If x_n is Cauchy,

$\exists N_1 \in \mathbb{N} : \forall n > N_1 \quad |x_{N_1+1} - x_n| < 1$

$\Rightarrow \quad |x_n| < |x_{N_1+1}| + 1$

This implies that $|x_n| \le K$, where

$K = \max\{|x_{N_1+1}| + 1, |x_1|, \dots |x_N|\},$

so every Cauchy sequence must be bounded.

Continued...

By Bolzano-Weierstrass, $\exists\, n_k$, $L \in \mathbb{R}$ such that

$$x_{n_k} \longrightarrow L, \quad k \to \infty$$

$\forall \varepsilon > 0$ take $N \in \mathbb{N}$ such that

$$\forall\, n, m > N \qquad |x_n - x_m| < \frac{\varepsilon}{2}$$

since $x_{n_k} \longrightarrow L$, take $n_k > N$ such that $|x_{n_k} - L| < \frac{\varepsilon}{2}$.

Then $\forall\, n > N$

$$|x_n - L| \le |x_n - x_{n_k}| + |x_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\Rightarrow \quad x_n \to L, \quad n \to \infty.$$

3. (10 points) (a) Show that the function $f : x \mapsto \frac{1}{x+1}$ is uniformly continuous on $[0, \infty)$.

$$|f(x) - f(y)| = \left| \frac{1}{x+1} - \frac{1}{y+1} \right|$$

$$= \frac{|x-y|}{|x+1|\,|y+1|} \leq |x-y|, \quad \text{if } x, y \geq 0.$$

$\forall \varepsilon > 0$ take $\delta = \varepsilon$, then $|x-y| < \delta \Rightarrow$

$|f(x) - f(y)| \leq |x-y| < \delta = \varepsilon$

$\Rightarrow f$ is uniformly continuous on $\{x : x \geq 0\}$.

(b) Give an example of a bounded continuous function on $(0, \infty)$ which is not uniformly continuous. Justify your answer.

$f(x) = \sin \frac{1}{x}$; $\quad -1 \leq f(x) \leq 1$,

f is continuous at every value $x > 0$,

however $x_n = \frac{1}{\pi n}$, $y_n = \frac{1}{\pi(2n + \frac{1}{2})}$

satisfy $|x_n - y_n| < \delta$ if n is large enough,

and $f(x_n) = 0$, $f(y_n) = 1$

$\Rightarrow |f(x_n) - f(y_n)| = 1.$

Continued...

Uniform continuity fails with $\varepsilon = 1$.

4. (10 points) Determine if the following sequences are convergent. If a sequence is divergent find a convergent subsequence.

(a) $x_n = \sum_{j=1}^{n} \frac{1}{j!}$

x_n is Cauchy:

$$|x_n - x_m| = \sum_{j=n+1}^{m} \frac{1}{j!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{m!}$$

$(m > n)$

$$\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{m-1}}$$

$$= \frac{1}{2^n}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{m-n-1}}\right)$$

$$= \frac{1}{2^n} \cdot 2\left(1 - \frac{1}{2^{m-n}}\right) \leq \frac{1}{2^{n-1}} < \varepsilon$$

if $m > n > N$, N is such that $2^{N-1} > \frac{1}{\varepsilon}$.

(b) $x_n = \sin\left(\frac{\pi n}{2} + \frac{1}{n}\right)$.
[Hint: the trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$ may be useful.]

$$x_n = \sin\frac{\pi n}{2}\cos\frac{1}{n} + \sin\frac{1}{n}\cos\frac{\pi n}{2}$$

$$\lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1.$$

$$-1 \leq \cos\frac{\pi n}{2} \leq 1, \quad -\left|\sin\frac{1}{n}\right| \leq \sin\frac{1}{n}\cos\frac{\pi n}{2} \leq \left|\sin\frac{1}{n}\right|$$

$$\Rightarrow \lim_{n\to\infty} \sin\frac{1}{n}\cos\frac{\pi n}{2} = 0$$

$\left\{\sin\frac{\pi n}{2}\right\} = \{1, 0, -1, 0, 1, \dots\}$ – divergent.

Convergent subsequences:

$n_k = 2k$, $n_k = 4k-3$, $n_k = 4k-1$

Continued...

5. (8 points) Consider the function

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Determine whether the derivatives $f'(0)$, $f''(0)$ exist; if they exist find their values.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^3 \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0$$

since $-h^2 \leq h^2 \sin \frac{1}{h} \leq h^2$.

for $x \neq 0$,

$$f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$$

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} =$$

$$= \lim_{h \to 0} \frac{3h^2 \sin \frac{1}{h} - h \cos \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} \left(3h \sin \frac{1}{h} - \cos \frac{1}{h}\right)$$ – Does not exist

(Pick $h_n = \frac{1}{\pi(2n + \frac{1}{2})} \Rightarrow \cos \frac{1}{h_n} = 1$, $\sin \frac{1}{h_n} = 0$

Pick $h_n' = \frac{1}{\pi(2n - \frac{1}{2})} \Rightarrow$
$\cos \frac{1}{h_n} = -1$, $\sin \frac{1}{h_n} = 0$.)

Continued...

6. (8 points) Prove, using the definition of the derivative that if u and v are differentiable at x_0 and $f(x) = u(x)\,v(x)$ then

$$f'(x_0) = u'(x_0)\,v(x_0) + u(x_0)\,v'(x_0).$$

$$f'(x_0) = \lim_{h \to 0} \frac{u(x_0+h)v(x_0+h) - u(x_0)v(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{u(x_0+h)v(x_0+h) - u(x_0+h)v(x_0) + u(x_0+h)v(x_0) - u(x_0)v(x_0)}{h}$$

$$= \lim_{h \to 0} u(x_0+h)\frac{v(x_0+h) - v(x_0)}{h}$$

$$+ \lim_{h \to 0} \frac{u(x_0+h) - u(x_0)}{h} v(x_0)$$

$$= u(x_0)\,v'(x_0) + u'(x_0)\,v(x_0).$$

Continued...

7. (bonus: 6 points) Give an example of a function satisfying the conditions of Rolle's theorem for which $f'(x) = 0$ for infinitely many values of x. Justify your answer.

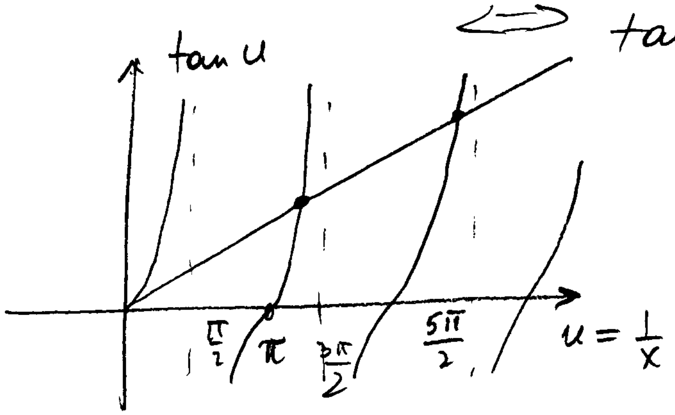
$$f(x) = \begin{cases} x \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$\mathcal{D}$ continuous on $[0, \frac{1}{\pi}]$, differentiable on $(0, \frac{1}{\pi})$;

$$f(0) = f(\tfrac{1}{\pi}) = 0$$

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} = 0$$

$\Longleftrightarrow \tan \frac{1}{x} = \frac{1}{x}$; has infinitely many solutions on $(0, \frac{1}{\pi})$:



($\tan u = u$ has infinitely many solutions on (π, ∞))

The end.