Wed, Mar 9, 2011 **MATH 350, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

CSUN ID No. : ____________________

This test includes 6 questions (48 points in total) in the main part, and one bonus question, worth 5 points. Please check that your copy of the test has 7 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (8 points) Prove that if $a \neq 0$ and $f(x) = 1/x$ then f is continuous at a.

$$|f(x) - f(a)| = \left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{|a||x|}$$

Now, if $|a| > 0$ and $\delta \le \frac{|a|}{2}$ then

$$|x-a| < \delta \Rightarrow |x-a| < \frac{|a|}{2} \Rightarrow |x| \ge \frac{|a|}{2}, \text{ so}$$

Then $\frac{|x-a|}{|a||x|} \le \frac{2}{a^2}|x-a| < \frac{2}{a^2}\delta \le \varepsilon$

implies $|f(x) - f(a)| < \varepsilon$, whenever

$|x-a| < \delta$, where $\delta = \min\left\{\frac{|a|}{2}, \frac{\varepsilon a^2}{2}\right\}$.

2. (8 points) (a) Using the axioms of an ordered field F prove that if $a, b \in F$ then $(-a)(-b) = ab$.

First, $(-a)b + ab = ((-a)+a)b = 0 \cdot b = 0$

(Since $(0+b)b = 0 \cdot b + b^2 = b^2$
$\Rightarrow 0 \cdot b = 0$)

which implies $(-a)b = -ab$

Using commutativity,

$a(-b) = -ab \Rightarrow (-a)(-b) = -(-ab) = ab$

(since $-c + c = 0 \Rightarrow$
$-(-c) = c$, by uniqueness of additive inverse.)

(b) Use part (a) to conclude that in every ordered field F, $1 > 0$.

$\{0\}$ is not a field $\Rightarrow 1 \neq 0$.

If $1 < 0$ then $(-1) > 0$

So $(-1)(-1) = 1 \cdot 1 = 1 > 0$, by the axiom of order
a contradiction with the hypothesis $1 < 0$.

Therefore, $1 > 0$.

3. (8 points) (a) State the definition of $\mathbb{N}$, the set of natural numbers.

$a \in \mathbb{R}$ is in $\mathbb{N}$ if a is an element of every inductive subset of $\mathbb{R}$.

(b) Prove by induction that if $m, n \in \mathbb{N}$ then $m + n \in \mathbb{N}$.

Define $S = \{n \in \mathbb{N} : \forall m \in \mathbb{N} \; m+n \in \mathbb{N}\}$.

Then $1 \in S$ since $\forall m \in \mathbb{N}$ $m+1 \in \mathbb{N}$ as $\mathbb{N}$ is inductive.

If $x \in S$ then $\forall m \in \mathbb{N}$

$m + (x+1) = (m+x)+1 \in \mathbb{N}$ since $m+x \in \mathbb{N}$

$\Rightarrow x+1 \in S$

Therefore S is inductive, and since $S \subseteq \mathbb{N}$ we must have $S = \mathbb{N}$.

Continued...

4. (8 points) Determine a value $\delta > 0$ so that for a given $\varepsilon > 0$ the statement "$|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$" is valid:

$$f(x) = \frac{\sin x}{x}, \quad a = 0, \quad L = 1.$$

[Hint: you may use the inequalities $x \cos x \leq \sin x \leq x$ valid for $-\pi/2 \leq x \leq \pi/2$.]

$$|f(x) - L| = \left| \frac{\sin x}{x} - 1 \right| = 1 - \frac{\sin x}{x}$$

$$\leq 1 - \cos x, \quad \text{if } |x| < \frac{\pi}{2}.$$

$$1 - \cos x < \varepsilon \iff \cos x > 1 - \varepsilon$$

$$\iff |x| < \arccos(1-\varepsilon), \text{ if } \varepsilon \leq 2$$

and $\iff x \in \mathbb{R}$ if $\varepsilon > 2$.

Thus, if $\varepsilon > 2$, $\delta = \frac{\pi}{2}$ is OK;

if $\varepsilon \leq 2$,

$\delta = \min\left\{ \frac{\pi}{2}, \arccos(1-\varepsilon) \right\}$ works.

5. (8 points) Prove (using an ε-δ argument) that if

$$f(x) \to 0, \quad x \to +\infty \quad \text{and} \quad g(x) \to +\infty, \quad x \to 1+$$

then $h(x) = f(g(x))$ satisfies $\lim_{x \to 1+} h(x) = 0$.

$g(x) \longrightarrow +\infty, \quad x \to 1+$

$\overset{\text{def.}}{\iff} \forall A > 0 \ \exists \delta_A > 0 : \ (x > 1) \wedge (x - 1 < \delta_A) \Rightarrow g(x) > A$

$f(x) \to 0, \quad x \to +\infty$

$\overset{\text{def.}}{\iff} \forall \varepsilon > 0 \ \exists A > 0 : \quad y > A \Rightarrow |f(y)| < \varepsilon$

Given $\varepsilon > 0$ choose A that works in the 2^{nd} definition.

For such A, take δ_A that works in the 1^{st} definition.

Then $(x > 1) \wedge (x - 1 < \delta_A) \Rightarrow |f(g(x))| < \varepsilon$

$\Rightarrow |h(x)| < \varepsilon.$

6. (8 points) Find all $p \in \mathbb{R}$ such that

$$\lim_{x \to +\infty} x^p \sin(1/x)$$

is finite. For those p, find the limit. Use the properties of limits to justify your answers.

Since $\lim_{u \to 0} \frac{\sin(u)}{u} = 1$, we note that

[in the case $p = 1$] $\lim_{x \to +\infty} x \sin\left(\frac{1}{x}\right) = \lim_{x \to +\infty} \frac{\sin(1/x)}{1/x} = 1.$

Thus,

$$x^p \sin\frac{1}{x} = x^{p-1}\, x \sin\frac{1}{x}$$

If $p > 1$ then $\lim_{x \to +\infty} x^{p-1} = +\infty$

[For, $x^{p-1} > A \iff x > A^{\frac{1}{p-1}}$]

$$\Rightarrow \lim_{x \to +\infty} x^p \sin\frac{1}{x} = \lim_{x \to +\infty} x^{p-1}\, x \sin\frac{1}{x} = +\infty \cdot 1 = +\infty.$$

If $p < 1$ then $p - 1 < 0$ and

$$\lim_{x \to +\infty} x^{p-1} = 0$$

[For, $x^{p-1} < \varepsilon \iff x > \varepsilon^{\frac{1}{p-1}}$]

Therefore,

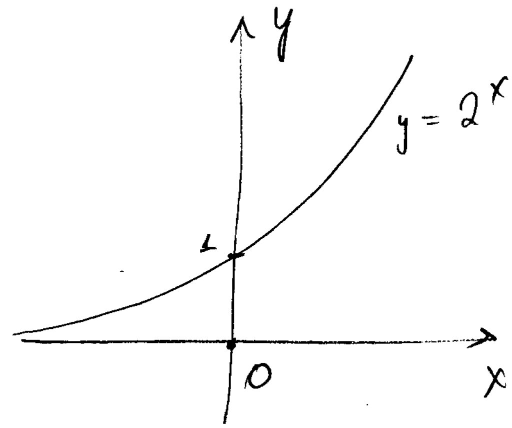
$$\lim x^p \sin\frac{1}{x} = \lim x^{p-1}\, x \sin\frac{1}{x} = 0 \cdot 1 = 0.$$

7. (extra credit: 5 points) Give an example of a <u>one-to-one</u> and <u>onto</u> function $f : A \to B$,

(a) if $A = \mathbb{R}$, $B = (0, \infty)$.

$f(x) = 2^x$;

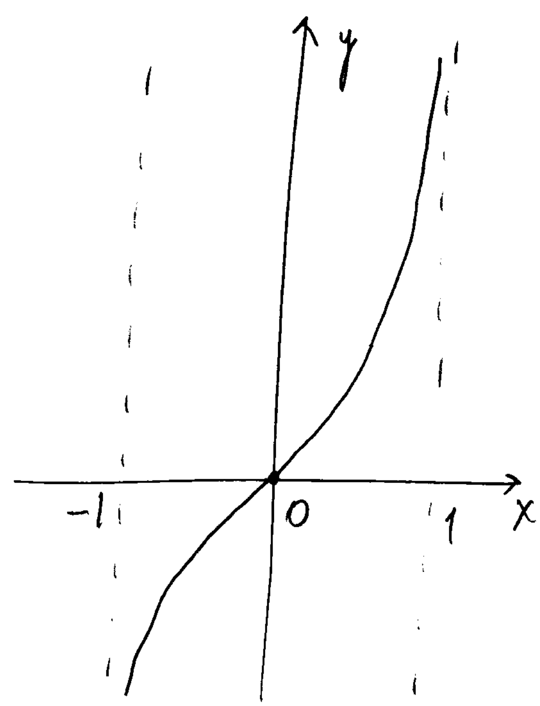
the range is $(0, \infty)$



(b) if $A = (-1, 1)$, $B = \mathbb{R}$.

$$f(x) = \frac{x}{1-|x|}$$

the range is $\mathbb{R}$.



The end.