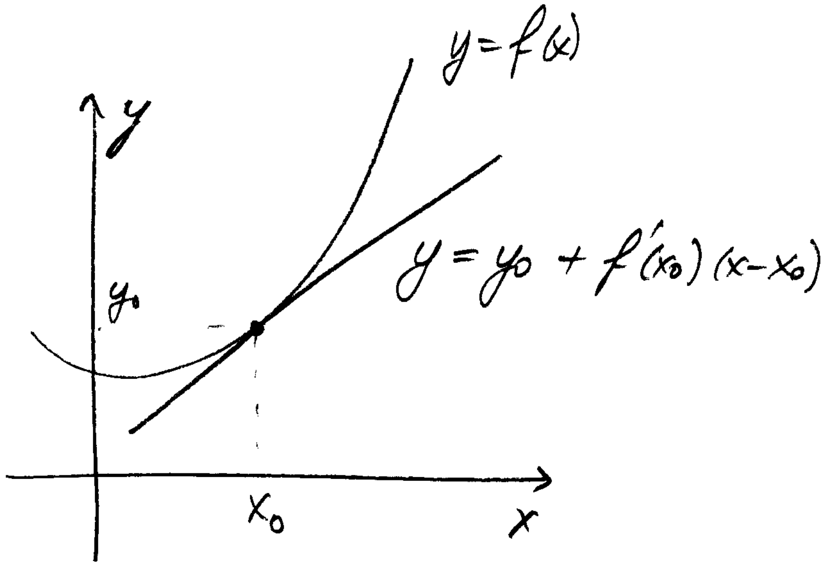
## Taylor's formula

The idea of differential calculus is that "regular" (differentiable) functions $f(x)$ are well approximated by their tangent lines for $x$ close to a reference point. The derivative $f'(x_0)$ gives the value of the slope of the graph at $x_0$.




The quantitative version of this statement is precisely the "Fundamental lemma of differentiation" that we used for the proof of Chain Rule.

<u>Lemma</u>: If $f$ is differentiable at $x_0$ then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + (x-x_0)\eta(x-x_0)$$

where $\eta(h) \to 0$ as $h \to 0$.

A natural idea is to try to refine this kind of approximation by using higher degree polynomials, say

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + A_2(x-x_0)^2 + o((x-x_0)^2)$$

Here's a formal way to obtain coefficients of such a higher degree polynomial.

$$f(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \dots + A_n(x-x_0)^n + o((x-x_0)^n),$$

plug in $x_0 \Rightarrow f(x_0) = A_0$.

$$f'(x) = A_1 + 2A_2(x-x_0) + \dots + nA_n(x-x_0)^{n-1} + o((x-x_0)^{n-1})$$

plug in $x_0 \Rightarrow f'(x_0) = A_1$

$$f''(x) = 2A_2 + 3\cdot 2\cdot A_3(x-x_0) + \dots + n(n-1)A_n(x-x_0)^{n-2} + o((x-x_0)^{n-2})$$

plug in $x_0 \Rightarrow f''(x_0) = 2A_2$

$$\vdots$$

$$f^{(k)}(x_0) = k!\,A_k, \text{ or}$$

$$A_k = \frac{f^{(k)}(x_0)}{k!}$$

Then the natural choice of the polynomial to use is

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

<u>Def</u>: The above polynomial is Taylor's polynomial for $f(x)$ about $x = x_0$.

We hope (so far, it's formal calculation) that

$$f(x) = p_n(x) + \underbrace{o((x-x_0)^n)}_{\text{means } (x-x_0)^n \eta(x-x_0)}.$$

where $\eta(h) \to 0$ as $h \to 0$.

(The above equ. is known as Taylor's formula)



<u>Def</u>: $r_n(x_0, x) = f(x) - p_n(x_0, x)$

is the Taylor remainder of order $n$.

<u>Theorem</u> (Lagrange's form for Taylor remainder.)

Let $n \in \mathbb{N}$ and let $f^{(n+1)}$ be continuous, differentiable on $(a,b) \ni x_0$. Then

for $\forall x \in (a,b)$, $r_n(x_0, x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$

for a certain $c$ between $x$ and $x_0$.

<u>Example</u>: $f(x) = e^x$; $f^{(j)}(x) = e^x$, $j \in \mathbb{N}$

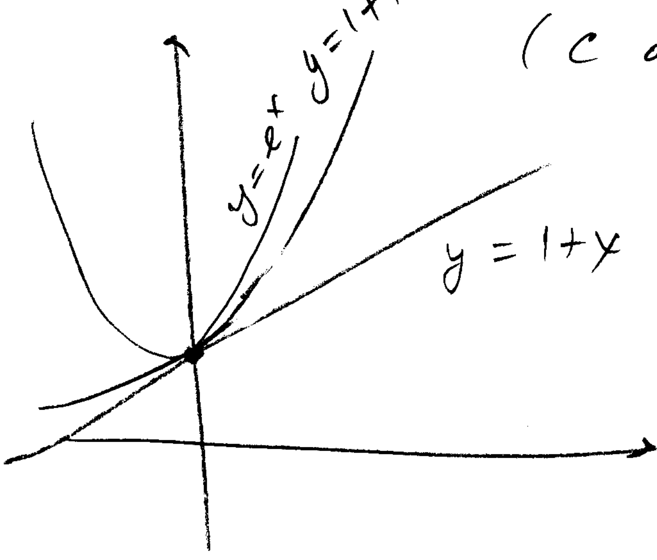
$x_0 = 0$; then $f^{(j)}(0) = 1$

$$p_n(x_0, x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}$$

$$f(x) = e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + r_n(x_0, x)$$

where $r_n(x_0, x) = \dfrac{e^c}{(n+1)!}(x - x_0)^{n+1}$

($c$ depends on $x$; is between $x_0 = 0$ and $x$)




If we fix $x$ and increase $n$, then

$$\frac{e^c x^{n+1}}{(n+1)!} \le \frac{e^x x^n}{(n+1)!}$$

Notice that $\forall x > 0 \quad \frac{x^n}{n!} \longrightarrow 0, \quad n \to \infty$

(Indeed, let $a$ be the smallest natural number $\geq x$, then $\forall n > a$,

$$\frac{x^n}{n!} \leq \frac{a^n}{n!} = \frac{a \cdot \ldots \cdot a}{1 \cdot 2 \ldots a} \frac{a \ldots a}{(a+1) \ldots n}$$

$$\leq a^a \cdot q^{n-a}, \text{ where}$$

$$q = \frac{a}{a+1} < 1. \Rightarrow \frac{a^a}{q^a} \cdot q^n \underset{n \to \infty}{\longrightarrow} 0.)$$

Thus, for every $x$ fixed $r_n(x_0, x) \longrightarrow 0$ as $n \to \infty$.

The approximation is visibly good for $x$ close to $x_0$; moreover the remainder conv. to zero for every $x$ fixed. If we keep $n$ fixed however, the approximation deteriorates as $x$ becomes large.

<u>Proof of Taylor's theorem.</u>

$n = 0$ is Lagrange's mean value theorem:

$$f(x) = f(x_0) + f'(c)(x - x_0).$$

For $n \geq 1$ WTS

$$f^{(n+1)}(c) = M(n+1)! \text{ where}$$

$$M = \frac{f(x) - p_n(x_0, x)}{(x - x_0)^{n+1}}.$$

Define $g : (a,b) \to \mathbb{R}$ by

$$g(t) = f(t) - p_n(x_0, t) - M(t - x_0)^{n+1}$$

Then $g^{(n+1)}(t) = f^{(n+1)}(t) - M(n+1)!$

$g(x_0) = 0, \quad g(x) = 0 \Rightarrow$

$g'(c_1) = 0;$

$g'(x_0) = \cdots = g^{(n)}(x_0) = 0.$

$\Rightarrow g''(c_2) = 0 \Rightarrow \cdots g^{(n+1)}(c_{n+1}) = 0.$ $\square$

# Integration by parts and Taylor's formula

**Thm** (Int. by parts.) $u, v : [a, b] \longrightarrow \mathbb{R}$ such that $u', v' : [a, b] \longrightarrow \mathbb{R}$ continuous. Then

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v$$

**Proof**:

$$uv' + u'v = (uv)' ;$$

$$\int_a^b (uv)' = [uv]_a^b = \int_a^b uv' + \int_a^b u'v \quad \boxed{\ }.$$

**Thm** Let $f$ be such that $f^{(n+1)}$ is continuous on the interval with end points $x_0$ and $x$; then

$$f(x) = p_n(x_0, x) + r_n(x_0, x), \text{ where}$$

$$r_n(x_0, x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)\,(x-t)^n\,dt ,$$

**Proof**: 
$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt = - \int_{x_0}^{x} f'(t)\,(x-t)'\,dt$$

$$= \int_{x_0}^{x} f''(t)\,(x-t)\,dt - \Big[f'(t)\,(x-t)\Big]_{x_0}^{x}$$

$$\Rightarrow f(x) = f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)\,(x-t)\,dt.$$

continuing,

$$f(x)-f(x_0)= f'(x_0)(x-x_0) - \int_{x_0}^{x} f''(t)\left(\tfrac{1}{2}(x-t)^2\right)' dt$$

$$= f'(x_0)(x-x_0) - \left[f''(t)\,\tfrac{1}{2}(x-t)^2\right]_{x_0}^{x}$$

$$+ \int_{x_0}^{x} f'''(t)\,\tfrac{1}{2}(x-t)^2 dt$$

$$= f'(x_0)(x-x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2$$

$$+ \tfrac{1}{2}\int_{x_0}^{x} f'''(t)\,(x-t)^2 dt$$

Induction step:

$$\frac{1}{(n-1)!}\int_{x_0}^{x} f^{(n)}(t)\,(x-t)^{n-1} dt$$

$$= -\frac{1}{n!}\int_{x_0}^{x} f^{(n)}(t)\,\left((x-t)^{n}\right)' dt$$

$$= -\frac{1}{n!}\left[f^{(n)}(t)\,(x-t)^n\right]_{x_0}^{x} + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)\,(x-t)^n dt$$

$$= \frac{1}{n!} f^{(n)}(x_0)\,(x-x_0)^n + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)\,(x-t)^n dt.$$

$\square$

<u>Corollary</u>: Lagrange's form of the remainder

$$r_n(x_0,x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)\,(x-t)^n dt$$

$$= \frac{1}{n!} f^{(n+1)}(c)\int_{x_0}^{x} (x-t)^n dt$$

$$= \frac{1}{n!} f^{(n+1)}(c)\left[\frac{-1}{n+1}(x-t)^{n+1}\right]_{x_0}^{x}$$

$$= \frac{1}{(n+1)!} f^{(n+1)}(c)\,(x-x_0)^{n+1}.$$ [Use Problem 16 in 5.1.]

## Uniqueness of the Taylor expansion

Suppose that we found

"local Taylor" (LT) $\quad f(x) = P_n(x_0, x) + o\left((x-x_0)^n\right)$

used again for $(x-x_0)^n$.
$\eta(x-x_0)$ where $\eta(h) \to 0$, $h \to 0$.

where

$$P_n(x_0, x) = A_0 + A_1(x-x_0) + \dots + A_n(x-x_0)^n$$

Is it true that $P_n(x_0, x) = p_n(x_0, x)$ (the Taylor polynomial)?

<u>Theorem</u>. Suppose $f^{(n+1)}$ is defined and continuous on an interval $(a,b) \ni x_0$, and that (LT) holds. Then $P_n(x_0, x)$ is necessarily the Taylor polynomial of $f$ at $x_0$.

<u>Rmk</u>: The conditions of this theorem are excessively strong; the same result holds under the sole assumption that $f(x_0) \dots f^{(n)}(x_0)$ are defined, and (LT) holds, but that is somewhat more difficult to prove.

<u>Proof</u>: We know $f(x) = A_0 + A_1(x-x_0) + \dots + A_n(x-x_0)^n + o(x-x_0)^n$.

$$\Rightarrow \lim_{x \to x_0} f(x) = A_0 ;$$

$$\frac{f(x)-A_0}{x-x_0} = A_1 + A_2(x-x_0) + \dots + A_n(x-x_0)^{n-1} + o(x-x_0)^{n-1}$$

$$\Rightarrow \quad A_1 = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

$$\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = A_2 + A_3(x-x_0) + \dots + A_n(x-x_0)^{n-2} + o(x-x_0)^{n-2}$$

$$\Rightarrow \quad A_2 = \lim_{x\to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2}$$

(know this limit exists!)

By induction,

$$A_k = \lim_{x\to x_0} \frac{f(x)-A_0-A_1(x-x_0)\dots - A_{k-1}(x-x_0)^{k-1}}{(x-x_0)^k}$$

Thus $A_0 \dots A_n$ are uniquely determined. $\square$

**Examples** 1) $f(x) = \sin x; \quad x_0 = 0$

$$f'(x) = \cos x = \sin\left(x+\frac{\pi}{2}\right)$$

$$f''(x) = -\sin x = \cos\left(x+\frac{\pi}{2}\right) = \sin(x+\pi)$$

$$f^{(k)}(x) = \sin\left(x+\frac{\pi k}{2}\right).$$

$$f^{(k)}(0) = \sin\left(\frac{\pi k}{2}\right) = 0, 1, 0, -1, 0 \dots$$

$$P_1(0,x) = x$$
$$P_3(0,x) = x - \frac{x^3}{6}$$
$$P_5(0,x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$
$$\vdots$$
$$P_{2n+1}(0,x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Then $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right)$

<u>Exercise</u>: Derive the formula

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right)$$

<u>Exercise</u>: Derive Lagrange's form of the remainder, and show that

for $\sin x$, $\left|r_n(0,x)\right| \leq \frac{1}{(n+1)!} |x|^{n+1}$

(this can be improved to next order for $n$ odd)

and for $\cos x$,

$$|r_n(0,x)| \leq \frac{1}{(n+1)!} |x|^{n+1}$$

(can be improved to next order if $n$ is even.)

2) $f(x) = \ln(1+x)$ , $x_0 = 0$

$$f'(x) = \frac{1}{1+x} \qquad f^{(3)}(x) = \frac{2}{(1+x)^3}$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f^{(4)}(x) = -\frac{2\cdot 3}{(1+x)^4}$$

$$\vdots$$

$$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}$$

$$f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

$$p_n(0,x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n-1}}{n}x^n$$

and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{(-1)^{n-1}}{n}x^n + o(x^n).$$

For $x > 0$ Lagrange's form of the remainder gives

$$|r_n(0,x)| = \frac{|(-1)^n|}{|(1+c)|^{n+1}} \frac{1}{n+1} |x|^{n+1} \leq \frac{|x|^{n+1}}{n+1}$$

For $x < 0$ it is somewhat more difficult to show that

Exercise $\rightarrow$ $|r_n(0,x)| \leq |x|^{n+1}$, $x \in (-1,0)$.

3) $f(x) = (1+x)^\alpha$ ; $\alpha \in \mathbb{R}$ ; $x_0 = 0$.

$$f^{(k)}(x) = \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k}$$

$$f^{(k)}(0) = \alpha(\alpha-1)\dots(\alpha-k+1)$$

Then

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + o(x^n).$$

<u>Problems</u> 1) Show that for $\ln(1+x)$

$$r_n(0,x) \le |x|^{n+1}, \quad |x|<1$$

2) Find Taylor polynomials of order $n$ for

$$(1-x^2)^{-\frac{1}{2}} \qquad x\ln(1+x)$$

$$\frac{1}{1-x} \qquad (x+1)\ln(1+x)$$

Find the first 5 terms for $\arcsin(x)$

3) Compute the values to 5 decimals of accuracy. Use Lagrange's form of the remainder to guarantee the result:

$$e^{0.2}; \quad \log(0.9); \quad (63)^{\frac{1}{6}}$$

4) Compute the integral to 5 decimals

$$\int_0^1 \frac{1-\cos x}{x}\,dx.$$