Taylor's formula

The idea of differential calculus is that "regular" (differentiable) functions $f(x)$ are well approximated by their tangent lines for $x$ close to a reference point. The derivative $f'(x_0)$ gives the value of the slope of the graph at $x_0$.

![Graph showing the approximation of $f(x)$ by $y_0 + f'(x_0)(x-x_0)$]

The quantitative version of this statement is precisely the "Fundamental Lemma of Differentiation" that we used for the proof of Chain Rule.

**Lemma:** If $f$ is differentiable at $x_0$ then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + (x-x_0)\gamma(x-x_0)$$

where $\gamma(h) \to 0$ as $h \to 0$.

A natural idea is to try to refine this kind of approximation by using higher degree polynomials, say

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + A_2(x-x_0)^2 + o((x-x_0)^2)$$
Here’s a formal way to obtain coefficients of such a higher degree polynomial:

\[ f(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots + A_n(x-x_0)^n + o((x-x_0)^n) \]

plug in \( x_0 \Rightarrow f(x_0) = A_0 \)

\[ f'(x) = A_1 + 2A_2(x-x_0) + \cdots + nA_n(x-x_0)^{n-1} + o((x-x_0)^{n-1}) \]

plug in \( x_0 \Rightarrow f'(x_0) = A_1 \)

\[ f''(x) = 2A_2 + 3\cdot 2A_3(x-x_0) + \cdots + n(n-1)A_n(x-x_0)^{n-2} + o((x-x_0)^{n-2}) \]

plug in \( x_0 \Rightarrow f''(x_0) = 2A_2 \)

\[ f^{(k)}(x_0) = k!A_k, \quad \text{or} \]

\[ A_k = \frac{f^{(k)}(x_0)}{k!} \]

Then the natural choice of the polynomial to use is

\[ p_m(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m \]

**Def.** The above polynomial is Taylor's polynomial for \( f(x) \) about \( x=x_0 \).

We hope (so far, it's formal calculation) that

\[ f(x) = p_m(x) + o((x-x_0)^{m+1}) \]

means \( (x-x_0)^m f(x-x_0) \) as \( h\to 0 \).
(The above eqn. is known as Taylor's formula)

**Definition**

\[ r_n(x_0, x) = f(x) - p_n(x_0, x) \]

is the Taylor remainder of order n.

**Theorem** (Lagrange's form for Taylor remainder)

Let \( n \in \mathbb{N} \) and let \( f^{(n+1)} \) be continuous, differentiable on \((a, b) \ni x_0 \). Then for \( \forall x \in (a, b) \),

\[ r_n(x_0, x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1} \]

for a certain \( c \) between \( x \) and \( x_0 \).

**Example**

\[ f(x) = e^x; \quad f^{(j)}(x) = e^x, \quad j \in \mathbb{N} \]

\( x_0 = 0 \); then \( f^{(j)}(0) = 1 \)

\[ p_n(x_0, x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \]

\[ f(x) = e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + r_n(x_0, x) \]

where \( r_n(x_0, x) = \frac{e^c}{(n+1)!} (x-x_0)^{n+1} \)

(\( c \) depends on \( x \); \( c \) between \( x_0 = 0 \) and \( x \))

If we fix \( x \) and increase \( n \), then

\[ \frac{e^c x^{n+1}}{(n+1)!} \leq \frac{e^x x^n}{(n+1)!} \]
Notice that \( \forall x > 0, \frac{x^n}{n!} \to 0, \, n \to \infty \)

(Indeed, let \( a \) be the smallest natural number \( \geq x \), then \( \forall n > a, \)

\[
\frac{x^n}{n!} \leq \frac{a^n}{n!} = \frac{a \ldots \cdot a}{1 \cdot 2 \ldots \cdot a} \cdot \frac{a \ldots \cdot a}{(a+1) \ldots \cdot n} \\
\leq a \cdot q^{n-a}, \\
q = \frac{a}{a+1} \leq 1. \Rightarrow \frac{a^a}{q^a} \cdot q^n \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus, for every \( x \) fixed \( r_n(x_0, x) \to 0 \)

as \( n \to \infty \).

The approximation is visibly good for \( x \) close to \( x_0 \); moreover the remainder converges to zero for every \( x \) fixed. If we keep \( n \) fixed however, the approximation deteriorates as \( x \) becomes large.

**Proof of Taylor's theorem.**

\( n=0 \) is Lagrange's mean value theorem:

\[ f'(c) = f(x_0) + f'(c)(x-x_0) \]

For \( n \geq 1 \) WTS

\[ f^{(n+1)}(c) = M(n+1)! \text{ where} \]

\[ M = \frac{f(x) - p_n(x_0, x)}{(x-x_0)^{n+1}} \]
Define \( g: (a,b) \to \mathbb{R} \) by
\[
g(t) = f(t) - p \alpha (x_0, t) - M (t - x_0)^{n+1}
\]
Then \( g^{(n+1)} (t) = f^{(n+1)} (t) - M (n+1)! \)
\( g(x_0) = 0, \quad g(x) = 0 \implies \)
\( g'(c_1) = 0; \)
\( g'(x_0) = \ldots = g^{(k)} (x_0) = 0. \)
\( \implies g'' (c_2) = 0 \implies \ldots g^{(n+1)} (c_{n+1}) = 0. \) \( \Box \)
Integration by parts and Taylor's formula

Then (Int. by parts.) \( u, v : [a, b] \to \mathbb{R} \)

such that \( u', v' : [a, b] \to \mathbb{R} \)
continuously. Then

\[
\int_a^b uv' = [uv]_a^b - \int_a^b u'v
\]

Proof:

\[
uv' + u'v = (uv)'
\]

\[
\int_a^b (uv)' = [uv]_a^b = \int_a^b uv' + \int_a^b u'v \tag{1}
\]

Then

Let \( f \) be such that \( f^{(n+1)} \) is continuous on the interval with end points \( x_0 \) and \( x \); then

\[
f(x) = p_n(x_0, x) + r_n(x_0, x)
\]

where

\[
r_n(x_0, x) = \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t) (x-t)^n dt
\]

Proof:

\[
f(x) - f(x_0) = \int_{x_0}^x f'(t) dt = - \int_{x_0}^x f'(t)(x-t) dt
\]

\[
= \int_{x_0}^x f''(t)(x-t) dt - \left[ f'(t)(x-t) \right]_{x_0}^x
\]

\[
=> f(x) = f(x_0) + f'(x_0)(x-x_0) + 
+ \int_{x_0}^x f''(t)(x-t) dt.
\]
continuing,

\[ f(x) - f(x_0) = f'(x_0)(x-x_0) - \int_{x_0}^{x} f''(t) \frac{1}{2} (x-t)^2 \, dt \]

\[ = f'(x_0)(x-x_0) - \left[ f''(t) \frac{1}{2} (x-t)^2 \right]_{x_0}^{x} + \int_{x_0}^{x} f'''(t) \frac{1}{2} (x-t)^2 \, dt \]

\[ = f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 \]

\[ + \frac{1}{2} \int_{x_0}^{x} f'''(t) (x-t)^2 \, dt \]

**Induction step:**

\[ \frac{1}{(n-1)!} \int_{x_0}^{x} f^{(n)}(t) (x-t)^{n-1} \, dt \]

\[ = -\frac{1}{n!} \int_{x_0}^{x} f^{(n)}(t) (x-t)^{n} \, dt \]

\[ = -\frac{1}{n!} \left[ f^{(n)}(t) (x-t)^{n} \right]_{x_0}^{x} + \frac{1}{n} \int_{x_0}^{x} f^{(n+1)}(t) (x-t)^{n} \, dt \]

\[ = \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^{n} + \frac{1}{n} \int_{x_0}^{x} f^{(n+1)}(t) (x-t)^{n} \, dt . \]

**Corollary:** Lagrange's form of the remainder

\[ r_n (x_0, x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t) (x-t)^{n} \, dt \]

\[ = \frac{1}{n!} f^{(n+1)}(c) \int_{x_0}^{x} (x-t)^{n} \, dt \]

\[ = \frac{1}{n!} f^{(n+1)}(c) \left[ \frac{-1}{n+1} (x-t)^{n+1} \right]_{x_0}^{x} \]

\[ = \frac{1}{(n+1)!} f^{(n+1)}(c) (x-x_0)^{n+1} . \] [Use Problem 16 in 5.1.]}
Uniqueness of the Taylor Expansion

Suppose that we found

\[ f(x) = P_n(x_0, x) + o((x-x_0)^n) \]

where

\[ P_n(x_0, x) = A_0 + A_1(x-x_0) + \ldots + A_n(x-x_0)^n \]

Is it true that \( P_n(x_0, x) = P_n(x_0, x) \)

(the Taylor polynomial)?

\[ \lim_{x 	o x_0} \eta(x) = \eta(x_0) \]

**Theorem.** Suppose \( f^{(n+1)} \) is defined and continuous on an interval \((a, b)\) containing \( x_0 \), and that (LT) holds. Then \( P_n(x_0, x) \) is necessarily the Taylor polynomial of \( f \) at \( x_0 \).

**Proof:** We know \( f(x) = A_0 + A_1(x-x_0) + \ldots + A_n(x-x_0)^n + o((x-x_0)^n) \)

\[ \lim_{x \to x_0} f(x) = A_0 \]
\[
\frac{f'(x)}{x-x_0} = A_1 + A_2 (x-x_0) + \cdots + A_n (x-x_0)^{n-1} + o((x-x_0)^{n-1})
\]

\[
\Rightarrow A_1 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x-x_0} = f'(x_0)
\]

\[
f(x) - f(x_0) - f'(x_0)(x-x_0) = A_2 + A_3 (x-x_0) + \cdots + A_n (x-x_0)^{n-2} + o((x-x_0)^{n-2})
\]

\[
\Rightarrow A_2 = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0)}{(x-x_0)^2}
\]

(Know this limit exists!)

By induction,

\[
A_k = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) \cdots - A_{k-1}(x-x_0)^{k-1}}{(x-x_0)^k}
\]

Thus \( A_0, \ldots, A_n \) are uniquely determined.

**Examples**

1) \( f(x) = \sin x; \quad x_0 = 0\)

\[
f'(x) = \cos x = \sin (x + \frac{\pi}{2})
\]

\[
f''(x) = -\sin x = \cos (x + \frac{\pi}{2}) = \sin (x + \pi)
\]

\[
f'''(x) = \sin x = \sin (x + \frac{3\pi}{2})
\]

\[
f''''(x) = \cos x = \sin (x + \pi + \frac{\pi}{2}) = \sin (x + 2\pi)
\]

\[
f(x) = \sin (x + \frac{\pi k}{2})
\]

\[
f'(0) = \sin (\frac{\pi k}{2}) = 0 \text{ for } k \neq 0, \pm 1, \ldots
\]
\[ p_1(0,x) = x \]
\[ p_3(0,x) = x - \frac{x^3}{6} \]
\[ p_5(0,x) = x - \frac{x^3}{6} + \frac{x^5}{120} \]
\[ p_{2n+1}(0,x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \]

Then \[ s_{2n} x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) \]

**Exercise:** Derive the formula

\[ \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1}) \]

**Exercise:** Derive Lagrange's form of the remainder, and show that

for \[ s_{2n} x, \quad \left| r_n(0,x) \right| \leq \frac{1}{(n+1)!} /x/^{n+1} \]

(this can be improved to next order for \( n \text{ odd} \))

and for \[ \cos x, \quad \left| r_n(0,x) \right| \leq \frac{1}{(n+1)!} /x/^{n+1} \]

(can be improved to next order if \( n \text{ is even} \))
2) \( f(x) = \ln(1 + x) \), \( x_0 = 0 \)

\[
\begin{align*}
\varphi'(x) &= \frac{1}{1+x} \quad & f^{(2)}(x) &= \frac{2}{(1+x)^3} \\
\varphi''(x) &= -\frac{1}{(1+x)^2} \quad & f^{(4)}(x) &= -\frac{2 \cdot 3}{(1+x)^4} \\
\varphi^{(k)}(x) &= (\frac{-1}{1+x})^{k-1}(k-1)! \\
f^{(k)}(0) &= (\frac{-1}{1})^{k-1}(k-1)! \\
\end{align*}
\]

\[
\begin{align*}
p_n(0, x) &= x^2 - \frac{x^3}{3} + \frac{x^4}{4} - \cdots + \frac{(-1)^{n-1}x^n}{n} \\
\ln(1 + x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + o(x^n). \\
\end{align*}
\]

For \( x > 0 \), Lagrange's form of the remainder gives

\[
|p_n(0, x)| = \frac{|(-1)^{n+1}|}{|1+x|^{n+1}} \leq \frac{|x|^{n+1}}{n+1}
\]

For \( x < 0 \), it is somewhat more difficult to show that

Exercise \( \rightarrow \) \( |p_n(0, x)| \leq |x|^{n+1} \) \( x \in (-1, 0) \).

3) \( f(x) = (1 + x)^d \); \( d \in \mathbb{R} \); \( x_0 = 0 \).

\[
\begin{align*}
\varphi^{(k)}(x) &= \frac{d(d-1) \cdots (d-k+1)(x)^{d-k}}{k!} \\
f^{(k)}(0) &= \frac{d(d-1) \cdots (d-k+1)}{k!}
\end{align*}
\]
Then
\[(1+x)^d = 1 + dx + \frac{d(d-1)}{2} x^2 + \ldots + \frac{d(d-1)\ldots (d-n+1)}{n!} x^n + o(x^n).\]

Problems

1) Show that for \( \ln(1+x) \)
\[v_n(0, x) \leq |x|^{n+1}, \quad |x| < 1\]

2) Find Taylor polynomials of order \( n \) for \( (1-x^2)^{-\frac{1}{2}} \times \ln(1+x) \)
\[
\frac{1}{1-x} \quad (x+1) \ln(1+x)
\]
Find the first 5 terms for \( \arctan(x) \)

3) Compute the values to 5 decimals of accuracy.
Use Lagrange's form of the remainder to guarantee the result:
\[e^{0.2} ; \log(0.9) ; (63)^{\frac{1}{2}} \]

4) Compute the integral to 5 decimals
\[
\int_0^1 \frac{1-\cos x}{x} \, dx.
\]