Homework 12, Problems 1-6

1. Use Taylor’s theorem to prove the binomial theorem: for \( n \in \mathbb{N} \):

\[
(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \cdots + x^n.
\]

2. Find a Taylor polynomial for the function \( e^x \) about \( x = 0 \) which approximates the function within the accuracy of \( 10^{-2} \) on the interval \([-1,1]\).

3. Derive the Taylor expansion

\[
\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n}).
\]

Show that the remainder satisfies the estimate

\[
|r_n(0, x)| \leq \frac{1}{(n+1)!}|x|^{n+1}
\]

(this can be improved to next order for \( n \) even).

4. Consider the function \( f(x) = \ln(1 + x) \), \( x_0 = 0 \), and let \( r_n(0, x) \) be the remainder of order \( n \) in Taylor’s formula. Show that \( |r_n(0, x)| \leq \frac{|x|^{n+1}}{(1+x)(n+1)} \) for \(-1 < x \leq 0\). [Hint: Use the integral form of the remainder.]

5. Find the Taylor polynomials of order \( n \) with \( x_0 = 0 \) for the functions

(a) \( f(x) = \frac{1}{\sqrt{1-x^2}} \)
(b) \( f(x) = \arcsin(x) \)
(c) \( f(x) = \frac{1}{1-x} \)
(d) \( f(x) = (1 + x) \ln(1 + x). \)

6. Let

\[
f(x) = \begin{cases} 
  e^{-1/x^2}, & x \neq 0 \\
  0, & x = 0
\end{cases}
\]

Find the Taylor polynomial of order \( n \) with \( x_0 = 0 \). [See also Problem 17, 9.4.]