Homework 12, Problems 1-6

1. Use Taylor's theorem to prove the binomial theorem: for $n \in \mathbb{N}$:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \cdots + x^n.$$

2. Find a Taylor polynomial for the function e^x about $x = 0$ which approximates the function within the accuracy of 10^{-2} on the interval $[-1, 1]$.

3. Derive the Taylor expansion

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n}).$$

Show that the remainder satisfies the estimate

$$|r_n(0,x)| \le \frac{1}{(n+1)!}|x|^{n+1}$$

(this can be improved to next order for n even).

4. Consider the function $f(x) = \ln(1+x)$, $x_0 = 0$, and let $r_n(0,x)$ be the remainder of order n in Taylor's formula. Show that $|r_n(0,x)| \le \frac{|x|^{n+1}}{(1+x)(n+1)}$ for $-1 < x \le 0$. [Hint: Use the integral form of the remainder.]

5. Find the Taylor polynomials of order n with $x_0 = 0$ for the functions

   (a) $f(x) = \frac{1}{\sqrt{1-x^2}}$

   (b) $f(x) = \arcsin(x)$

   (c) $f(x) = \frac{1}{1-x}$

   (d) $f(x) = (1+x)\ln(1+x)$.

6. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Find the Taylor polynomial of order n with $x_0 = 0$. [See also Problem 17, **9.4**.]