Wed, Nov 24, 2010 **MATH 350, Midterm 2** Prof. V. Panferov

Name: (print) ______________________________

CSUN ID No. : Solutions.

This test includes 7 questions (56 points in total), on 7 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (8 points) Prove that if b is a boundary point of A and $b \notin A$ then b is a limit point of A. Give an example of a set A and a boundary point b which is not a limit point of A. State the necessary definitions.

<u>Def 1</u>: $b \in \mathbb{R}$ is a boundary pt of A if
$\forall \delta > 0 \;\; \exists x \in A, \; \exists x' \notin A : \;\; x, x' \in (b-\delta, b+\delta)$

<u>Def 2</u>: $b \in \mathbb{R}$ is a limit pt. of A if
$\forall \delta > 0 \;\; \exists x \in A, \;\; x \neq b : \;\; x \in (b-\delta, b+\delta)$.

Since b is a bdry pt of A, $\forall \delta > 0 \; \exists x \in A$, $x \in (b-\delta, b+\delta)$.
Since $b \notin A$ and $x \in A$, $x \neq b$.
Therefore b is a limit pt. of A.

<u>Examples</u>: Any isolated pt of A is a bdry pt.
$\{x_1, x_2 \ldots x_n\}$ — any pt. is a bdry pt, not a limit pt.
$\mathbb{Z}$; any pt. is a bdry pt. $[-1, -\frac{1}{2}) \cup \{0\} \cup (\frac{1}{2}, 1]$ — $b = 0$ is a bdry pt.

2. (8 points) Show that if $f, g : (a,b) \to \mathbb{R}$ are continuous at a point $x_0 \in (a,b)$ and $g(x_0) \neq 0$ then f/g is continuous at x_0.

Using sequences: $\forall x_n \to x_0$

$g(x_n) \neq 0$ if n is sufficiently large (since g is continuous at x_0)

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim f(x_n)}{\lim g(x_n)} = \frac{f(x_0)}{g(x_0)}$$

$\Rightarrow$ f/g is continuous at x_0.

Using ε-δ:

$$\left|\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}\right| = \left|\frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}\right|$$

$$= \left|\frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}\right|$$

$$\leq \frac{1}{|g(x)|}|f(x) - f(x_0)| + \frac{|f(x_0)|}{|g(x_0)||g(x)|}|g(x) - g(x_0)|.$$

Since $g(x)$ is continuous at x_0 and $g(x_0) \neq 0$

$\exists \delta_1 > 0 : |g(x) - g(x_0)| < \frac{|g(x_0)|}{2}$

$\Rightarrow |g(x)| > \frac{|g(x_0)|}{2} = c_0$ if $|x - x_0| < \delta_1$.

Since f, g are continuous, Take $\delta_2 > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon c_0}{2}$

$\delta_3 > 0$ such that $|g(x) - g(x_0)| < \frac{\varepsilon c_0}{2} \frac{|g(x_0)|}{|f(x_0)|}$

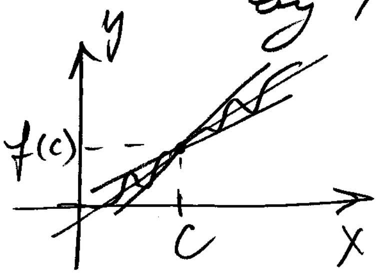
Then take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

Continued...

$$\Rightarrow \left|\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ if } |x - x_0| < \delta.$$

3. (8 points) Prove that if a function $f:(a,b)\to\mathbb{R}$ has a relative extremum at $c\in(a,b)$ and f is differentiable at c then $f'(c)=0$. Give an example of a function $f:(a,b)\to\mathbb{R}$ which has a relative extremum at c but $f'(c)\neq 0$.

<u>Proof 1</u>: By contradiction. If $f'(c)\neq 0$, then by Theorem 5.5 $\exists\,\delta>0$: $f(x)>f(c)$ for $(x-c)f'(c)>0$



and $f(x)<f(c)$ for $(x-c)f'(c)<0$, if $|x-c|<\delta$.

This contradicts the assumption that c is a point of rel. extremum.

<u>Proof 2</u>: Direct:

If c is a point of rel. max,

$$\frac{f(x)-f(c)}{x-c}\geq 0 \text{ if } |x-c|<\delta,\ x>c$$

and $$\frac{f(x)-f(c)}{x-c}\leq 0 \text{ if } |x-c|<\delta,\ x<c$$

$$\Rightarrow \lim_{x\downarrow c}\frac{f(x)-f(c)}{x-c}=f'(c)\geq 0$$

and $$\lim_{x\uparrow c}\frac{f(x)-f(c)}{x-c}=f'(c)\leq 0.$$

$$\Rightarrow f'(c)=0.$$

4. (8 points) Show that if f is uniformly continuous on (a, b) and $x_n \in (a, b)$ is a Cauchy sequence then $f(x_n)$ is a Cauchy sequence. [Recall that x_n is Cauchy if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$: $m, n > N \;\Rightarrow\; |x_n - x_m| < \varepsilon$.]

f is unif. continuous :

$\forall \varepsilon > 0 \;\; \exists \delta > 0 \; : \;\; |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$

x_n is Cauchy $\Rightarrow$ for $\delta > 0$ as above

$\exists N_\delta \in \mathbb{N} : \;\; n, m > N_\delta \Rightarrow |x_n - x_m| < \delta$.

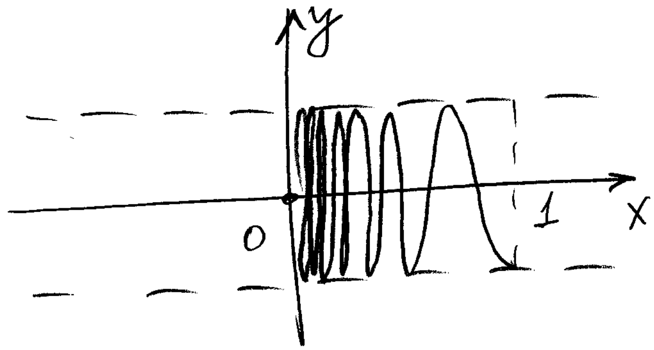
Combining the above 2 statements we obtain

$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} : \;\; n, m > N$

$\Rightarrow |f(x_n) - f(x_m)| < \varepsilon$

$\Rightarrow f(x_n)$ is Cauchy.

5. (8 points) Prove that the function $f(x) = \cos(\frac{\pi}{x})$ is not uniformly continuous on $(0, 1)$.



Idea: as $x_0 \to 0$, the value of δ required to satisfy the cond. of continuity for a given ε will become smaller and smaller.

(using the prev. problem)

Practical solution: take $x_n \to 0$;

(then x_n is Cauchy, since any convergent sequence is Cauchy.)

in such a way that $f(x_n) = \cos\frac{\pi}{x_n}$ diverges (is not Cauchy.)

For example: Solve $\cos\frac{\pi}{x} = 0$

$$\Rightarrow \frac{\pi}{x} = \frac{\pi}{2} + \pi n$$

$$\Rightarrow x = \frac{1}{n+\frac{1}{2}}, \quad n \in \mathbb{Z}$$

Solve $\cos\frac{\pi}{x} = 1$

$$\Rightarrow \frac{\pi}{x} = 2\pi n; \quad x = \frac{1}{2n}, \quad n \in \mathbb{Z}$$

Take $x_n = \begin{cases} \frac{1}{n+\frac{1}{2}}, & n\text{ - even} \\ \frac{1}{2n}, & n\text{ - odd} \end{cases}$

Then $x_n \to 0 \Rightarrow x_n$ is Cauchy, but

$\cos\frac{\pi}{x_n} = (1, 0, 1, 0, \ldots)$ diverges $\Rightarrow$ not Cauchy.

Continued...

6. (8 points) Find the Taylor polynomial of order 5 for the function $f(x) = \log(1+x)$ about $x = 0$. Estimate the error of approximation of $\log(2)$ when using this polynomial.

$f(x) = \log(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f^{(3)}(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$\vdots$

$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}$ $\qquad f^{(n)}(0) = (-1)^{n+1}(n-1)!$

$$p_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

To approximate $\log(2)$ use $x = 1$.

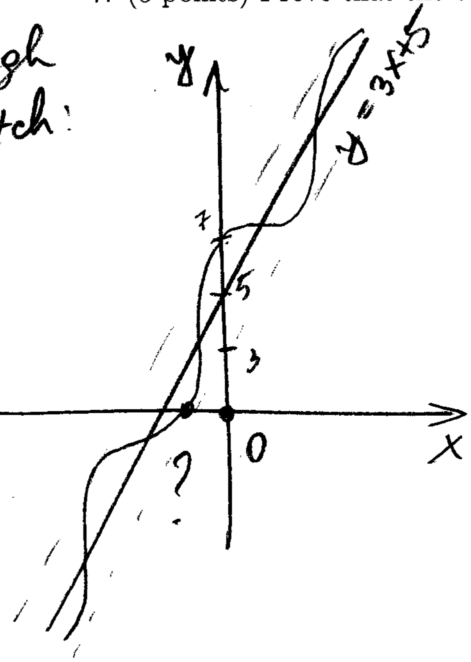
$$\log 2 = \log(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + r_5(0,1).$$

$$r_5(0,1) = \frac{f^{(6)}(c)}{6!} 1^{n+1}$$

$$\Rightarrow |r_5(0,1)| = \frac{1}{6} \frac{1}{(1+c)^6} \le \frac{1}{6}.$$

7. (8 points) Prove that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root.

Rough Sketch:



It is clear that

$f(x) = 3x + 2\cos x + 5$

satisfies $f(x) > 0$ for $x > 0$, large

and $f(x) < 0$ for $x < 0$, large.

(e.g. $f(10) = 35 + 2\cos 10 > 33$,
and $f(-10) = -25 + 2\cos 10 < -23$)

By intermediate value theorem on $[-10, 10]$ there exists at least one value $x \in (-10, 10)$ such that $f(x) = 0$.

Assume $f(x) = 0$ has 2 solutions $x = x_1$ and $x = x_2$.
(say $x_2 > x_1$)

Then $f(x_1) = f(x_2)$

$\Rightarrow$ by Rolle's theorem $\exists\, c \in (x_1, x_2)$: $f'(c) = 0$.

However, $f'(x) = 3 - 2\sin x \geq 1 > 0$

for any $x \in \mathbb{R}$

$\Rightarrow$ contradiction.