Mon, Oct 11, 2010 **MATH 350, Midterm 1** Prof. V. Panferov

Name: (print) ______________________

CSUN ID No. : Solutions.

This test includes 6 questions (44 points in total), on 6 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (6 points) Give an example of a convergent sequence that is not monotone increasing or decreasing. Give an example of a sequence that diverges to $+\infty$ and is not monotone.

(a) $x_n = (-1)^n \frac{1}{n}$

$\left(-1, +\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots\right)$

$x_n \to 0, \quad n \to \infty.$

(b) $x_n = (2+(-1)^n)n$

$(1, 6, 3, 12, 5, 18, 7, \dots)$

$x_n \geq n \Rightarrow x_n \to \infty, \quad n \to \infty.$

2. (8 points) Prove that every monotone increasing sequence a_n that is bounded above is convergent.

Let $a_n \leq C$, $a_{n+1} \geq a_n$, $\forall n \in \mathbb{N}$.

Take $L = \sup a_n$; show that $a_n \to L$.

By def. of supremum,

$$a_n \leq L,$$

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : \ a_{n_\varepsilon} > L - \varepsilon.$$

Since $\forall n \geq n_\varepsilon \quad a_n \geq a_{n_\varepsilon}$

we have

$$\forall n \geq n_\varepsilon \qquad L - \varepsilon < a_n \leq L$$

$$\Rightarrow \quad -\varepsilon < a_n - L \leq 0$$

$$\Rightarrow \quad |a_n - L| < \varepsilon.$$

We showed by the ε-δ-definition of the limit that $\lim_{n \to \infty} a_n = L$.

3. (8 points) If $f : X \to Y$ is a one-to-one function and A, B are two subsets of X prove that $f(A \cap B) = f(A) \cap f(B)$. Give an expample of a function f (not one-to-one) and sets A, B for which the above equality does not hold.

If $y \in f(A \cap B)$ then $\exists x \in A \cap B$: $f(x) = y$.

Since $x \in A$ we have $y \in f(A)$

Since $x \in B$, $y \in f(B)$

$\Rightarrow y \in f(A) \cap f(B)$ by definition of intersection.

If $y \in f(A) \cap f(B)$ then $\exists x_1 \in A$: $f(x_1) = y$

and $\exists x_2 \in B$: $f(x_2) = y$.

Since f is one-to-one, $x_1 = x_2 = x$

and $x \in A \cap B$.

$\Rightarrow y \in f(A \cap B)$.

We proved double inclusion, therefore

$f(A \cap B) = f(A) \cap f(B)$.

4. (6 points) Using the axioms of an ordered field F prove that if $a \in F$, $a \neq 0$ then $a^2 > 0$. Use this to deduce that in any ordered field $1 > 0$.

If $a \neq 0$ then either $a \in P$ or $-a \in P$.

If $a \in P$ then $a^2 = a \cdot a \in P$ (since P is closed under multiplication) (P is the set of positive numbers)

If $a \in P$, then $-a \in P$, and

$(-a) \cdot (-a) = a \cdot a$

Indeed, since $a \cdot a + (-a) \cdot a$

$= (a + (-a)) \cdot a = 0 \cdot a = 0$,

$-(a \cdot a) = -a \cdot a$

but $(-a) \cdot (-a) + (-a) \cdot a = (-a) \cdot ((-a) + a)$

$= (-a) \cdot 0 = 0$

So by uniqueness of additive inverse

$(-a) \cdot (-a) = a \cdot a$

Since $(-a) \cdot (-a) \in P$

we have $a^2 = a \cdot a \in P$

which proves that $a^2 \in P$ whenever $a \neq 0$.

To show that $1 > 0$

write $1 = 1 \cdot 1$ and use the previous argument.

Continued...

5. (8 points) Define a sequence (a_n) by $a_1 = 1$, $a_{n+1} = \sqrt{2a_n}$, $n = 1, 2 \ldots$

   (a) Show that $a_n \leq 2$.

   (b) Show that a_n is monotone increasing.

   (c) Find $\lim a_n$.

(a) Clearly $a_1 \leq 2$. By induction, if $a_n \leq 2$, then

$$a_{n+1} = \sqrt{2 \cdot a_n} \leq \sqrt{2 \cdot 2} = 2$$

$\Rightarrow$ induction step follows.

(b) $$\frac{a_{n+1}}{a_n} = \frac{\sqrt{2a_n}}{a_n} = \frac{\sqrt{2}}{\sqrt{a_n}} \geq \frac{\sqrt{2}}{\sqrt{2}} = 1$$

(c) $\lim a_n$ exists since a_n is bounded and monotone increasing.

Since $a_{n+1}^2 = 2a_n$,

$$\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} a_{n+1} = a$$

$$\Rightarrow a^2 = 2a$$

$$\Rightarrow a = 0 \text{ or } a = 2$$

Since $a_n \geq 1$ $a = 0$ is impossible

$$\Rightarrow a = 2.$$

6. (8 points) Consider the sequence $a_n = \frac{2}{n^2} + \cos(\pi n + \frac{1}{n})$. Find the upper and lower limits of a_n. Find subsequences of a_n that converge to the upper and lower limits.

$$a_n = \frac{2}{n^2} + \underbrace{\cos(\pi n)}_{\substack{\parallel \\ (-1)^n}} \cdot \cos\frac{1}{n} - \underbrace{\sin(\pi n)}_{\substack{\parallel \\ 0}} \sin\frac{1}{n}$$

$$= \frac{2}{n^2} + (-1)^n \cos\frac{1}{n}.$$

If n is even,

$$a_n = \frac{2}{n^2} + \cos\frac{1}{n} \to 1, \quad n \to \infty$$

If n is odd,

$$a_n = \frac{2}{n^2} - \cos\frac{1}{n} \to -1, \quad n \to \infty$$

$\Rightarrow$ 1 and -1 are the only possible subsequential limits,

$\Rightarrow$ $\limsup a_n = 1$,

$\liminf a_n = -1$.

Subsequences $a_{2K} \to 1$

$a_{2K+1} \to -1$.