#9 (HW8)

The Taylor polynomial $p_n(0, x) = 0$, $\forall n \in \mathbb{N}$.

Using Lagrange's form for the remainder,

$\forall x \in (-1, 1)$

$$|f(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x|^{n+1} \leq C^{n+1} |x|^{n+1}$$
$$= |Cx|^{n+1}$$

If $|Cx| < 1$, i.e. $x \in (-\frac{1}{C}, \frac{1}{C})$, then let $n \to \infty$ in

$$|f(x)| \leq |Cx|^{n+1}$$
$$\Rightarrow |f(x)| \leq 0, \quad x \in (-\tfrac{1}{C}, \tfrac{1}{C})$$
$$\Rightarrow f(x) = 0, \quad x \in (-\tfrac{1}{C}, \tfrac{1}{C}).$$

If $C < 1$ then we ~~are~~ done, otherwise take any $x_0 \in (-\frac{1}{C}, \frac{1}{C})$, use Taylor's formula centered at x_0 to obtain

$$f(x) = 0, \quad x \in (x_0 - \tfrac{1}{C}, x_0 + \tfrac{1}{C}).$$

Since $x_0 \in (-\frac{1}{C}, \frac{1}{C})$ is arbitrary,

$\Rightarrow f(x) = 0, \quad x \in (-\frac{2}{C}, \frac{2}{C})$.

Proceed by induction to cover the interval $(-1, 1)$ by finitely many intervals $(-\frac{k}{C}, \frac{k}{C})$.