Mon, Jul 7, 2015 MATH 350 Quiz 8

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. If $h(x) = f_1(x) + f_2(x)$ for $x \in [a, b]$ show that

$$\overline{\int_a^b} h(x)\,dx \le \overline{\int_a^b} f_1(x)\,dx + \overline{\int_a^b} f_2(x)\,dx.$$

$M_i(h) \ge M_i(f_1) + M_i(f_2)$, since $\sup(f_1+f_2) \le \sup f_1 + \sup f_2$.

$\forall P \subseteq I$ - partition

$$S^+(h, P) \geq S^+(f_1, P) + S^+(f_2, P)$$

Fix P, and let $P' \supseteq P$ be a refinement.

Then

$$S^+(h, P') \geq S^+(f_1, P') + S^+(f_2, P)$$

$$\Rightarrow \inf_{P'} S^+(h, P') \geq \left[\inf_{P'} S^+(f_1, P')\right] + S^+(f_2, P).$$

$$\Rightarrow \overline{\int_a^b} h \geq \overline{\int_a^b} f_1 + S^+(f_2, P)$$

Take inf over P:

$$\inf S^+(f_2, P) \geq \overline{\int_a^b} h - \overline{\int_a^b} f_1$$

$$\Rightarrow \overline{\int_a^b} f_2 \geq \overline{\int_a^b} h - \overline{\int_a^b} f_1$$

$$\Rightarrow \overline{\int_a^b} h \leq \overline{\int_a^b} f_1 + \overline{\int_a^b} f_2.$$

Please turn over...

2. Suppose that f and g are positive and continuous on the interval $[a, b]$. Prove that there is a number $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

Let $M = f(x_1) = \max_{[a,b]} f(x)$, $m = f(x_2) = \min_{[a,b]} f(x)$.

Then $g(x) \geq 0 \Rightarrow \quad m\,g(x) \leq f(x)g(x) \leq M g(x)$

$$\Rightarrow \quad m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

If $\int_a^b g(x)dx = 0$, this shows that $\int_a^b f(x)g(x)dx = 0$

$\Rightarrow$ any $\xi \in [a,b]$ works.

If $\int_a^b g(x) > 0$, then

$$f(x_1) = m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M = f(x_2)$$

By IVT $\exists\, \xi \in [a,b] : f(\xi) = \dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \Rightarrow \int_a^b fg = f(\xi)\int_a^b g.$

3. (a) Suppose that f is continuous on $[a, b]$. Show that all upper and lower Darboux sums are Riemann sums.

$$S^-(f,P) = \sum_{i=1}^n m_i(f)\Delta x_i = \sum_{i=1}^n f(x_i^*)\Delta x_i \quad \text{– Riemann}$$

By EVT $m_i = f(x_i^*)$

$$S^+(f,P) = \sum_{i=1}^n M_i(f)\Delta x_i = \sum_{i=1}^n f(x_i^{**})\Delta x_i \quad \text{– Riemann}$$

By EVT $M_i = f(x_i^{**})$

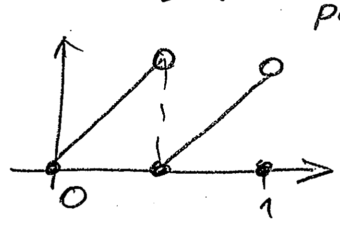
(b) Give an example of a bounded function defined on $[a, b]$ for which a Darboux sum is not a Riemann sum.

Any fn. where sup or inf are not achieved; for instance:

$(a, b) = [0,1]$

$f(x) = x - \dfrac{[2x]}{2}$

where $[x]$ is the integer part of x.



$P = (0, \frac{1}{2}, 1)$

$S^+(f,P) = 1\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 1$

However, any Riemann sum

$S(f,P,X) = \sum_{n=1}^n f(x_i)\Delta x_i < 1$

since $f(x_i) < 1$.