Thu, Jul 2, 2015 **MATH 350** **Quiz 7**

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. If f is increasing on the interval $I = [a, b]$ prove that f is integrable.

Let $P = (t_0, t_1 \ldots t_n)$, $t_0 = a$, $t_n = b$ be any partition of I.

f increasing $\Rightarrow m_i(f) = f(t_{i-1})$, $M_i(f) = f(t_i)$.

$$\Rightarrow S^+(f,P) - S^-(f,P) = \sum_{i=1}^{n} \big(\underbrace{f(t_i) - f(t_{i-1})}_{\geq 0}\big)\Delta x_i$$

$$\leq \sum_{i=1}^{n} \big(f(t_i) - f(t_{i-1})\big) \cdot \underbrace{\max_{i=1..n} \Delta x_i}_{=:\Delta x} = \Delta x \big(f(t_n) - f(t_0)\big)$$

$$= \Delta x \big(f(b) - f(a)\big) < \varepsilon$$

if $\Delta x < \dfrac{\varepsilon}{f(b) - f(a)}$ $\Rightarrow$ f is integrable by the Criterion of integrability

2. If f and g are nonnegative bounded functions on $I = [a, b]$, and $M = \sup_I f$, $m = \inf_I f$, $N = \sup_I g$, $n = \inf_I g$, show that

$$\sup f(x)\, g(x) - \inf f(x)\, g(x) \leq MN - mn.$$

$\begin{matrix} f(x) \leq M \\ g(x) \leq N \end{matrix} \Rightarrow f(x)g(x) \leq MN \Rightarrow \sup f(x)g(x) \leq MN$

(since $f(x) \geq 0$, $g(x) \geq 0$) (sup is the least upper bd; MN is an upper bd.)

$\begin{matrix} f(x) \geq m \\ g(x) \geq n \end{matrix} \Rightarrow f(x)g(x) \geq mn \Rightarrow \inf f(x)g(x) \geq mn$

(since $f(x), g(x) \geq 0$, $m, n \geq 0$) (inf is the greatest l.b. mn is a l.b.)

$\Rightarrow \sup f(x)g(x) - \inf f(x)g(x) \geq MN - mn.$

(properties of inequalities)

Please turn over...

3. Suppose f and g are nonnegative, bounded, and integrable on $I = [a, b]$. Prove that fg is integrable on I. Hint: use the result of the previous problem.

Take $\varepsilon > 0$. Since f_1 is integrable, $\exists P_1$, $S^+(f, P_1) - S^-(f, P_2) < \frac{\varepsilon}{N}$

Since g is int. $\exists P_2$, $S^+(g, P_2) - S^-(g, P_2) < \frac{\varepsilon}{M}$.

Let $P = P_1 \cup P_2$ — common refinement.

$$\text{then } S^+(fg, P) - S^-(fg, P) \le \sum_{i=1}^{n} (M_i N_i - m_i n_i) \Delta x_i$$

$$= \sum_{i=1}^{n} ((M_i - m_i) N_i + m_i (N_i - n_i)) \Delta x_i$$

$$\le N \sum_{i=1}^{n} (M_i - m_i) \Delta x_i + M \sum_{i=1}^{n} (N_i - n_i) \Delta x_i$$

$$< N \cdot \frac{\varepsilon}{2N} + M \frac{\varepsilon}{2M} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

4. (bonus: 2 points) Give an example of a function f defined on $[0, 1]$ such that $|f|$ is integrable but f is not.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & \text{otherwise}. \end{cases}$$

Then $\forall P \subseteq [0,1]$, partition, $\forall i$ $m_i(f) = \min_{I_i} f = -1$, $M_i(f) = \max_{I_i} f = 1$;

$$S^+(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = 1 \cdot \sum_{i=1}^{n} \Delta x_i = 1$$

$$S^-(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = (-1) \sum_{i=1}^{n} \Delta x_i = -1$$

$$-1 = \underline{\int_a^b} f \ne \overline{\int_a^b} f = 1$$

$\Rightarrow f$ is not integrable.