Wed, Jun 24, 2015 MATH 350 Quiz 6

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Show that the sequence x_n is Cauchy:

$$x_n = \sum_{j=1}^{n} \frac{(-1)^j}{j!}.$$

Let $n > m$, then

$$X_n - X_m = \sum_{j=m+1}^{n} \frac{(-1)^j}{j!} = \frac{(-1)^{m+1}}{(m+1)!} + \frac{(-1)^{m+2}}{(m+2)!} + \dots + \frac{(-1)^n}{n!}$$

if m even, $\frac{-1}{(m+1)!} \le X_n - X_m \le 0$

if m odd, $0 \le X_n - X_m \le \frac{1}{(m+1)!}$

$$\Rightarrow \quad |X_n - X_m| \le \frac{1}{(m+1)!} \xrightarrow[m \to \infty]{} 0$$

The sequence is Cauchy.

2. Determine whether or not the following sequence is Cauchy. Justify your answer.

$$x_n = (1 + (-1)^n)n + \frac{1}{n}.$$

No! $X_{2k} = 2 \cdot 2k + \frac{1}{2k} \ge 4k \xrightarrow[k \to \infty]{} +\infty$

the sequence is divergent $\Rightarrow$ is not Cauchy.

OR! $|X_n - X_m| = \left| (1+(-1)^n)n + \frac{1}{n} - (1+(-1)^m)m - \frac{1}{m} \right|$

let m - even, $n > m$ - odd

then $1+(-1)^n = 2$, $1+(-1)^m = 0$

$\Rightarrow |X_n - X_m| = \left| 2n + \frac{1}{n} - \frac{1}{m} \right| \ge 2n - 1 \ge A$

for any A, if n is large enough.

Please turn over...

3. Suppose $f : x \mapsto x^3$, $x_0 = 2$ in the Fundamental lemma of differentiation. Show that $\eta(h) = 6h + h^2$.

$$(x_0+h)^3 - x_0^3 = 3x_0^2 h + \eta(h)h$$

$$\cancel{x_0^3} + \cancel{3x_0^2 h} + 3x_0 h^2 + h^3 - \cancel{x_0^3} = \cancel{3x_0^2 h} + \eta(h)h$$

$$\eta(h) = 3x_0 h + h^2$$

$$\eta(h) = 6h + h^2.$$

4. Is the function $f(x)$ differentiable at $x_0 = 0$? Justify your answer.

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} \sin \frac{1}{h}$$ Does not exist,

since $\sin \frac{1}{h_n} = 0$ for $h_n = \frac{1}{\pi n} \to 0$

$\sin \frac{1}{h_n} = 1$ for $h_n = \frac{1}{\pi(2n+\frac{1}{2})} \to 0$

$\Rightarrow$ f is not differentiable at $x_0 = 0$