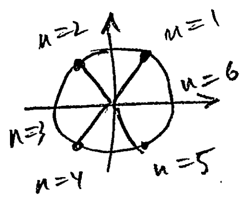
Mon, Jun 22, 2015 MATH 350 Quiz 5

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Determine if the given sequence converges to a limit. (Justify your answer.) In case the limit does not exist give an example of at least one convergent subsequence:

$$x_n = \left(1 - \frac{1}{n}\right) \sin \frac{\pi n}{3}.$$



$$\sin \frac{\pi n}{3} = \begin{cases} \frac{\sqrt{3}}{2}, & n = 1, 2, 7, 8, 13, 14 \dots \quad \text{i.e.} \left(\frac{3}{2} \pm \frac{1}{2}\right) + 6n \\ 0, & n = 3, 6, 9, \dots \\ -\frac{\sqrt{3}}{2}, & n = 4, 5, 10, 11, 16, 17 \quad \text{i.e.} \left(\frac{9}{2} \pm \frac{1}{2}\right) + 6n \end{cases}$$

$$1 - \frac{1}{n} \xrightarrow[n \to \infty]{} 1$$

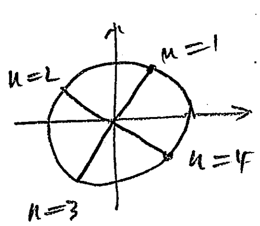
$$\left.\begin{array}{lll} n_k = \left(\frac{3}{2} \pm \frac{1}{2}\right) + 6k & \Rightarrow & x_{n_k} \to \frac{\sqrt{3}}{2} \\ n_k = 3k & \Rightarrow & x_{n_k} \to 0 \\ n_k = \left(\frac{9}{2} \pm \frac{1}{2}\right) + 6k & \Rightarrow & x_{n_k} \to -\frac{\sqrt{3}}{2}. \end{array}\right\} \Rightarrow \lim x_n \text{ D.N.E.}$$

2. Give an example of a sequence which has subsequences converging to 4 distinct real numbers. Justify your answer.

Ex: $(x_n) = (1, 2, 3, 4, 1, 2, 3, 4, 1, 2 \dots)$

$$x_{4k+1} = 1 \to 1, \quad n \to \infty$$
$$x_{4k+2} = 2 \to 2, \quad n \to \infty$$
$$x_{4k+3} = 3 \to 3, \quad n \to \infty$$
$$x_{4k} = 4 \to 4, \quad n \to \infty$$

Ex! $x_n = \sin\left(\frac{\pi n}{2} - \frac{\pi}{8}\right)$



$$n_k = 4k+1 \Rightarrow x_{n_k} = \sin\left(\frac{3\pi}{8}\right)$$
$$n_k = 4k+2 \Rightarrow x_{n_k} = \sin\left(\frac{\pi}{8}\right)$$
$$n_k = 4k+3 \Rightarrow x_{n_k} = -\sin\left(\frac{3\pi}{8}\right)$$
$$n_k = 4k \Rightarrow x_{n_k} = -\sin\left(\frac{\pi}{8}\right).$$

Please turn over...

3. Suppose that f is continuous on $[0, \infty)$ and that f is bounded. Give an example to show that the Extreme Value Theorem does not hold.

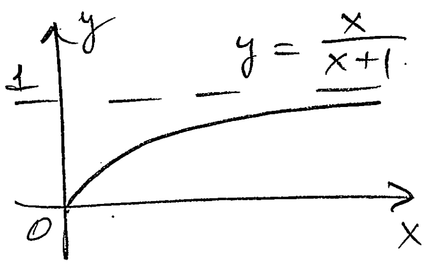
$f(x)$ is bounded, since $0 \le \frac{x}{x+1} \le 1$.

Ex: $f(x) = \frac{x}{x+1}$; Then $\sup\limits_{[0,\infty)} f(x) = 1$

Since $\frac{x}{x+1} \le 1$, $x \ge 0$, and $\frac{x}{x+1} > 1-\varepsilon \iff x > \frac{1}{\varepsilon} - 1$ is satisfied for x large enough

However $1 \notin f([0,\infty))$

since equation $\frac{x}{x+1} = 1$



$\iff x = x+1$

$\iff 0 = 1$ has no solution.

$\Rightarrow f(x)$ has no maximum on $[0, \infty)$.

4. Show that the linear function $f(x) = ax + b$ is uniformly continuous on $\mathbb{R}$.

$$|f(x_1) - f(x_2)| = |(ax_1 + b) - (ax_2 + b)|$$
$$= |a||x_1 - x_2| < |a|\delta = \varepsilon$$

if $|x_1 - x_2| < \delta$ and $\delta = \frac{\varepsilon}{|a|}$, $a \neq 0$.

On the other hand, if $a = 0$,

$|f(x_1) - f(x_2)| = |b - b| = 0$,

so any $\delta > 0$ works for any $\varepsilon > 0$.