Thu, Jun 16, 2015 **MATH 350** Quiz 4

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Find the limits of sequences, using ε-N technique, or theorems about limits:

   (a) $\sqrt{\dfrac{2}{n-1}}$ (b) $\dfrac{5^n}{n^n}$.

ε-N:

$\forall \varepsilon > 0$, $\left|\sqrt{\frac{2}{n-1}}\right| < \varepsilon$

$\iff \frac{2}{n-1} < \varepsilon^2$

$\iff n-1 > \frac{2}{\varepsilon^2}$

$\iff n > \frac{2}{\varepsilon^2}+1$

Pick $N > \frac{2}{\varepsilon^2}+1$ by Arch.P.

$\Rightarrow \left|\sqrt{\frac{2}{n-1}}\right| < \varepsilon$ for all $n > N$.

Squeeze Principle:

$\frac{5^n}{n^n} = \left(\frac{5}{n}\right)^n \leq \left(\frac{5}{n}\right)^5$, $n \geq 5$.

(for, $n \geq 5 \Rightarrow \frac{5}{n} \leq 1$.)

$\lim \left(\frac{5}{n}\right)^5 = \left(5 \lim \frac{1}{n}\right)^5$

$= (5 \cdot 0)^5 = 0$

Since $\lim \frac{1}{n} = 0$

and $x \mapsto x^5$ is continuous

$0 \leq \frac{5^n}{n^n} \leq \left(\frac{5}{n}\right)^5 \to 0$
$n \geq 5$

$\Rightarrow \lim \frac{5^n}{n^n} = 0$.

2. Give an example of a non-constant continuous function for which the Intermediate Value Theorem holds and such that there are infinitely many values x_0 such that $f(x_0) = c$.

in the formulation of the Theorem!

Let $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Then f is continuous at 0 (by the Squeeze Principle), satisfies assumptions of the Intermediate Value Theorem on $[a,b] = \left[-\frac{2}{3\pi}, \frac{2}{\pi}\right]$, $f(a) = -\frac{2}{3\pi} < 0 < \frac{2}{\pi} = f(b)$

and $f(x) = 0$

for infinitely many values $x = \frac{1}{\pi n}$, $n = 1, \pm 2, \pm 3 \ldots$

Please turn over...

3. Let $f : [a, b] \to \mathbb{R}$ be strictly increasing and continuous, and let $c = f(a)$, $d = f(b)$.

(a) Show that f is one-to-one.

(b) Use the intermediate value property to argue that $f([a, b]) = [c, d]$.

(a) f strictly increasing:

$\forall x_1, x_2 \in [a, b) \quad x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.

Show that f is one-to-one.

Let $x_1 \neq x_2$. Then either $x_1 > x_2$, so $f(x_1) > f(x_2)$

or $x_1 < x_2$, so $f(x_1) < f(x_2)$.

In either case $f(x_1) \neq f(x_2)$

$\Rightarrow f$ is one-to-one.

(b) Let $y \in [c, d]$.

If $y = c$ then $y = f(a)$, if $y = d$ then $y = f(b)$

$\Rightarrow y \in f([a, b])$.

If $c < y < d$

Int. Value Thm

we have $f(a) < y < f(b) \Rightarrow \exists x \in [a, b]$, $y = f(x)$.

$\Rightarrow y \in f([a, b])$.

Thus, $[c, d] \subseteq f([a, b])$.

On the other hand, since

$\forall x \in [a, b] \quad c = f(a) \leq f(x) \leq f(b) = d$

we have $f([a, b]) \subseteq [c, d]$

Therefore $f([a, b]) = [c, d]$.