Thu, Jun 18, 2015 MATH 350 Quiz 4.01

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Find the limits of sequences; give proofs using ε-N technique, or theorems about limits:

(a) $\dfrac{(-1)^n + \frac{1}{n}}{\frac{1}{n^2} - (-1)^n} = x_n$

$$x_n = \frac{(-1)^n\left((-1)^n + \frac{1}{n}\right)}{(-1)^n\left(\frac{1}{n^2} - (-1)^n\right)} = \frac{1 + \frac{(-1)^n}{n}}{\frac{(-1)^n}{n^2} - 1}$$

$$\lim x_n = \frac{1 + \lim \frac{(-1)^n}{n}}{\lim \frac{(-1)^n}{n^2} - 1} = \frac{1+0}{0-1} = -1$$

since

$$\underset{\searrow 0}{-\frac{1}{n}} \le \frac{(-1)^n}{n} \le \underset{\searrow 0}{\frac{1}{n}}, \quad \underset{\searrow 0}{-\frac{1}{n^2}} \le \frac{(-1)^n}{n^2} \le \underset{\searrow 0}{\frac{1}{n^2}}$$

Rmk: One could also look at two subsequences $n = 2k$, $n = 2k+1$, since any n is either even, or odd.

(b) $(3 + \sin n)n. \ge (3 + (-1))n = 2n$

since $\sin n \ge -1$.

$\forall A \in \mathbb{R}$ let $N > \frac{A}{2}$

$\Rightarrow (3 + \sin n)n > 2n > A$ for all $n > N$.

$\Rightarrow \lim (3 + \sin n)n = +\infty$

2. Give an example of a non-constant continuous function $f : [a, b] \to \mathbb{R}$ for which the Intermediate Value Theorem holds and such that there are infinitely many values x_0 in the formulation of the Theorem such that $f(x_0) = c$.

$$f(x) = \begin{cases} x\cos\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$\lim\limits_{x \to 0} f(x) = 0$ by Squeeze Princ.

$\Rightarrow f(x)$ is continuous.

$[a, b] = [-1, 1]$

$f(-1) = -\cos 1$; $f(1) = \cos 1$

Since $f(-1) < 0 < f(1)$

conditions of the I.V.T. are satisfied.

$f(x) = 0$ for $x = \dfrac{1}{\pi(n+\frac{1}{2})}$, $n \in \mathbb{Z}$

— infinitely many values on $[-1, 1]$.

Please turn over...

3. Let $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ be a polynomial of even degree. If $a_m a_0 < 0$ show that the equation has at least two real roots. What can be said in the case $a_m a_0 > 0$?

Assume $a_m a_0 < 0$, then $a_m \neq 0 \Rightarrow$

$$\text{Let } g(x) = x^m + \frac{a_{m-1}}{a_m} x^{m-1} + \dots + \frac{a_1}{a_m} x + \frac{a_0}{a_m}$$

$$=: x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0.$$

Then $b_0 < 0 \Rightarrow g(0) < 0$.

Also

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x^m \underbrace{\left(1 + \frac{b_m}{x} + \dots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^n}\right)}_{\xrightarrow{x\to\infty} 1}$$

$$= +\infty$$

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} x^m \underbrace{\left(1 + \frac{b_m}{x} + \dots + \frac{b_0}{x^m}\right)}_{\xrightarrow{x\to-\infty} 1}$$

$$= +\infty \quad (m\text{-even!})$$

Therefore $\exists\, x_1 < 0 \quad g(x_1) > 0$

$\exists\, x_2 > 0 \quad g(x_2) > 0$

Use IVT on $[x_1, 0]$ and on $[0, x_2]$

$\Rightarrow \exists\, x_0^{(1)} \in (x_1, 0) \quad \exists\, x_0^{(2)} \in (0, x_2)$

s.t. $g(x_0^{(1)}) = g(x_0^{(2)}) = 0$

$\Rightarrow g(x) = 0$ has at least 2 real solutions.

$\Rightarrow$ so does $f(x) = 0$.

If $a_m a_0 > 0$ the argument no longer holds. and one can find examples with or without real roots:

Ex: $f(x) = 1 + x^2$; $f(x) = 1 - 2x + x^2$; $f(x) = 1 - 3x + x^2$

no real roots. ; one real root ; two real roots. $(f(1) < 0)$