#5 (a) $f(x)=x^n$, $n\in N$

(defined inductively $x^1=x$, $x^{k+1}=x\,x^k$)

Show $x_1<x_2 \Rightarrow x_1^n<x_2^n$

$x_1,x_2\geq 0$

$n=1$ – clear

Let $x_1<x_2 \Rightarrow x_1^k<x_2^k$

Then $x_1^{k+1}=x_1x_1^k<x_1x_2^k<x_2x_2^k=x_2^{k+1}$.

(b) n – even, $n=2k$

$(-x)^{2k}=x^{2k}$ (by induction)

n – odd, $n=2k+1$

$(-x)^{2k+1}=(-x)^{2k}(-x)=-x^{2k+1}$

(c) $f(x)$ is unbounded above:

$\forall A\in\mathbb{R}$ $f(x)>A$

if $x>\max\{1,A\}$:

$f(x)=x^n>1>A$ if $A<1$

$f(x)=x^n>A^n\geq A$ if $A\geq 1$

(d) Let $n=2k+1$.

$\forall A\in\mathbb{R}$ $f(x)<A$

if $x<-\max\{1,|A|\}$

$f(x)=x^{2k+1}<(-1)<A$ if $A>-1$

$f(x)=x^{2k+1}<-|A|^{2k+1}\leq A$ if $A\leq -1$

(e) n is odd, $n = 2k+1$.

$$x_1 < x_2 < 0 \Rightarrow \quad 0 < -x_2 < -x_1$$
$$f(-x_2) < f(-x_1)$$
$$-f(x_2) < -f(x_1)$$
$$f(x_2) > f(x_1)$$
$$\Rightarrow f(x_1) < f(x_2).$$

If $x_1 < 0 < x_2 \Rightarrow f(x_1) < 0 < f(x_2)$

$\Rightarrow f(x)$ is str. increasing on $\mathbb{R}$.

$\forall c \in \mathbb{R}$ c is not an upper or a lower bound of $f(\mathbb{R})$

$$\Rightarrow \exists\, a, b : \quad f(a) < c < f(b)$$

$$\Rightarrow \exists\, x_0 \in \mathbb{R} : f(x_0) = c$$

x_0 is unique since $x > x_0 \Rightarrow f(x) > c$
$x < x_0 \Rightarrow f(x) < c$.

(Change the notations $c \leftrightarrow x$; $x \leftrightarrow y$ to obtain the result.)

(f) Use 4(c).