1

#5 (3.1)

f is increasing: $\forall x_1, x_2 \quad x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.

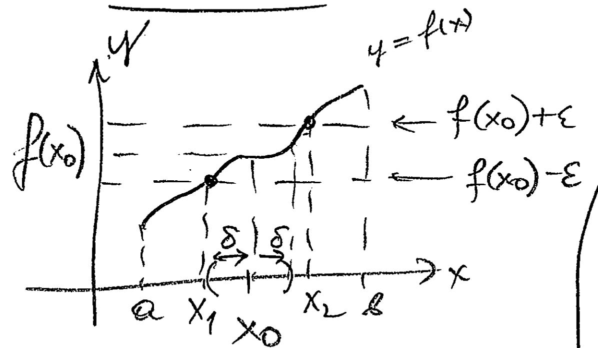
$f : [a, b] \to \mathbb{R}$

f satisfies the intermediate value property:

$\forall c \in (f(a), f(b)) \ \exists x_0 \in (a, b), \ f(x_0) = c$.

Prove that f is continuous on $[a, b]$.

Solution: Let $x_0 \in (a, b)$, $f(x_0) \in (f(a), f(b))$.



Let $\varepsilon > 0$.

If $f(x_0) - \varepsilon \in (f(a), f(b))$ let $x_1 \in (a, b)$ be such that $f(x_1) = f(x_0) - \varepsilon$; otherwise let $x_1 = a$.

If $f(x_0) + \varepsilon \in (f(a), f(b))$ let $x_2 \in (a, b)$ be such that $f(x_2) = f(x_0) + \varepsilon$, otherwise let $x_2 = b$.

See figure; [x_1, x_2 exist by the intermediate value property.]

Since f is increasing,

$$\forall x \quad x_1 < x < x_2 \Rightarrow \underset{f(x_0)-\varepsilon}{\underset{\parallel}{f(x_1)}} < f(x) < \underset{f(x_0)+\varepsilon}{\underset{\parallel}{f(x_2)}}$$

$$\Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

Take $\delta = \min\{x_2 - x_0, x_0 - x_1\}$

$$\Rightarrow \quad |x - x_0| < \delta \Rightarrow \quad x_1 < x < x_2 \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

If $x_0 = a$

let $\delta = \begin{cases} x_2 - a, & f(x_2) = f(x_0) + \varepsilon, \text{ if } f(x_0) + \varepsilon \in (f(a), f(b)) \\ b - a, & \text{otherwise} \end{cases}$

Then $a < x < a + \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$

If $x_0 = b$

let $\delta = \begin{cases} b - x_1, & f(x_2) = f(x_0) - \varepsilon, \text{ if } f(x_0) - \varepsilon \in (f(a), f(b)) \\ b - a, & \text{otherwise.} \end{cases}$

Then $b - \delta < x < b \Rightarrow |f(x) - f(a)| < \varepsilon.$