#12 (a) (5.1) $f : [a,b] \to \mathbb{R}$ – bounded; $I = [a,b]$

$$M = \sup_I f, \quad m = \inf_I f$$

$$M^* = \sup_I |f|, \quad m^* = \inf |f|$$

Solution: $\forall x, y \in I \quad f(x) \le M, \quad f(y) \ge m.$ $(-f(y) \le -m)$

$$\Rightarrow \quad f(x) - f(y) \le M - m$$

Similarly, $f(y) - f(x) \le M - m$

$$\Rightarrow \quad |f(x) - f(y)| \le M - m$$

$(|f(x)-f(y)| = \max\{f(x)-f(y), f(y)-f(x)\})$

by triangle inequality,

$$\forall x, y \in I \quad |f(x)| - |f(y)| \le |f(x) - f(y)| \le M - m.$$

Fix $y \in I$, take sup over x

$$\Rightarrow \quad \underbrace{\left(\sup_I |f(x)|\right)}_{= M^*} - |f(y)| \le M - m$$

$$\Rightarrow \forall y \in I \quad |f(y)| \ge M^* - M + m$$

Take inf over $y \in I$:

$$\Rightarrow \quad \inf_I |f(y)| \ge M^* - M + m$$

$$\Rightarrow \quad m^* \ge M^* - M + m$$

$$\Rightarrow \quad M^* - m^* \le M - m.$$