#10 (5.2)

Given $f: [a,b] \to \mathbb{R}$, $\int_a^b f^3 dx$ exists.
Show that $\int_a^b f dx$ exists. (1)

Solution

WTS: $\forall \varepsilon > 0 \ \exists P \in I$-partition,

$$\sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \varepsilon .$$

Know: $$\sum_{i=1}^{n} (M_i^3 - m_i^3)\Delta x_i < \varepsilon$$ (Notice that $M_i(f^3) = M_i^3$, $m_i(f^3) = m_i^3$.)

$$M_i - m_i = \frac{M_i^3 - m_i^3}{M_i^2 + M_i m_i + m_i^2} \le \frac{2}{M_i^{*2}} (M_i^3 - m_i^3)$$

where $M_i^* = \sup_{I_i} |f|$.

Indeed, $$M_i^2 + M_i m_i + m_i^2 = \frac{1}{2}(M_i^2 + 2M_i m_i + m_i^2) + \frac{1}{2}M_i^2 + \frac{1}{2}m_i^2$$
$$\ge \frac{1}{2}M_i^2 + \frac{1}{2}m_i^2 \ge \frac{1}{2}M_i^{*2},$$ since $M_i^{*2} = \max\{M_i^2, m_i^2\}$.

Let i_k denote all indices such that
$$M_{i_k}^* < \frac{\varepsilon}{4(b-a)} .$$

Then $$\sum_k (M_{i_k} - m_{i_k})\Delta x_{i_k} < \sum_k (M_{i_k}^* + M_{i_k}^*)\Delta x_{i_k}$$
$$\le \frac{\varepsilon}{2(b-a)} \sum_k \Delta x_{i_k} \le \frac{\varepsilon}{2} .$$

For the rest of the sum,

$$\sum_{i \ne i_k} (M_i - m_i)\Delta x_i \le \sum_{i \ne i_k} \frac{2}{M_i^{*2}} (M_i^3 - m_i^3)\Delta x_i$$
$$\le 2\left(\frac{4(b-a)}{\varepsilon}\right)^2 \sum_{i \ne i_k} (M_i^3 - m_i^3)\Delta x_i$$
$$\le 2\left(\frac{4(b-a)}{\varepsilon}\right)^2 \left(S^+(f^3, P) - S^-(f^3, P)\right)$$

Choose P so that

$$S^+(f^3,P) - S^-(f^3,P) < \frac{\varepsilon^3}{64(b-a)^2}.$$

Then

$$S^+(f,P) - S^-(f,P) = \sum_k (M_{i_k} - m_{i_k})\Delta x_{i_k}$$

$$+ \sum_{i \neq i_k} (M_i - m_i)\Delta x_i < \frac{\varepsilon}{2} + 2\left(\frac{4(b-a)}{\varepsilon}\right)^2 \cdot \frac{\varepsilon^3}{64(b-a)^2}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$