Mon, Jun 29, 2015 **MATH 350, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

CSUN ID No. : ______

This test includes 8 questions (54 points in total) in the main part, and two bonus questions, worth an extra 12 points. Please check that your copy of the test has 10 pages. The duration of the test is 1 hour 15 minutes.

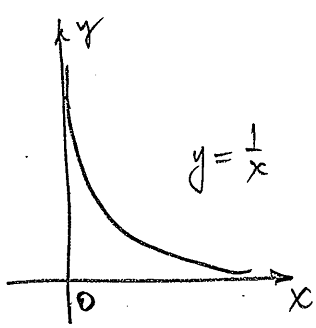
Your scores: (do not enter answers here)

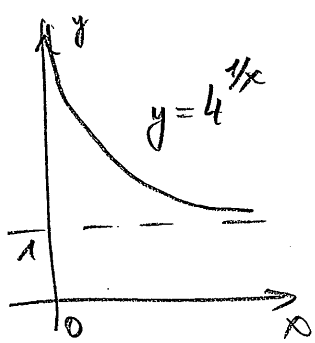
1	2	3	4	5	6	7	8	9	10	total

Important: The test is closed books/notes. No electronic devices are permitted except a basic scientific calculator. Show all your work.

1. (6 points) Let $f(x) = 4^{1/x}$, $x > 0$. Find the values $\sup\limits_{(0,\infty)} f$ and $\inf\limits_{(0,\infty)} f$ and justify your answers based on properties of supremums and infimums.

$x > 0 \Rightarrow 1/x > 0 \Rightarrow 4^{1/x} > 1$ ($\exp_4$ is increasing)





$y = 1$ is a lower bound;

Take $x_n = n$

$\Rightarrow \lim 4^{1/x_n} = \lim 4^{1/n} = 1$

$\Rightarrow 1$ is a greatest lower bound

$(\forall \varepsilon > 0\ \exists n:\ 4^{1/n} < 1 + \varepsilon)$

Take $x_n = \frac{1}{n}$; $4^{1/x_n} = 4^n \xrightarrow[n\to\infty]{} \infty$.

$\Rightarrow 4^{1/x}$ is unbounded on $(0, \infty)$, $\sup 4^{1/x} = +\infty$

$(\forall A \in \mathbb{R}\ \exists n:\ 4^{1/x_n} = 4^n > A)$

$\Rightarrow$ no upper bd in $\mathbb{R}$.

2. (8 points) Find the limits of sequences, using properties of limits. Show all steps.

(a) $\sqrt{n^2+n}-n$

$$\sqrt{n^2+n}-n = \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$$

$$= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$\lim_{n\to\infty} \sqrt{n^2+n}-n = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$\underset{\text{(lim of a cont. fn)}}{=} \frac{1}{\sqrt{1+\lim \frac{1}{n}}+1} = \frac{1}{2}$$

(b) $\left(\frac{2n+1}{2n-1}\right)^n$

$$\frac{2n+1}{2n-1} = \frac{2n-1+2}{2n-1} = 1+\frac{2}{2n-1}$$

$$= 1+\frac{1}{n-\frac{1}{2}}$$

$$\lim_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n-\frac{1}{2}}\right)^n$$

$$= \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n-\frac{1}{2}}\right)^{n-\frac{1}{2}}}_{=\,e} \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n-\frac{1}{2}}\right)^{-\frac{1}{2}}}_{=\,1}$$

since $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

1 since $\lim \frac{1}{n-\frac{1}{2}} = 0$, $(1+x)^{-\frac{1}{2}}$ is continuous at 0.

Thus

$$\lim_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n = e$$

Continued...

3. (6 points) Find the derivative of $f(x)$ at $x = x_0$, based on the definition:

$$f(x) = \frac{1}{1+x}, \quad x_0 = 3.$$

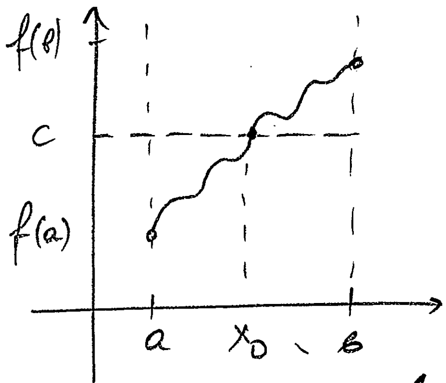
$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x \to 3} \frac{\frac{1}{1+x} - \frac{1}{1+3}}{x-3}$$

$$= \lim_{x \to 3} \frac{4-1-x}{4(1+x)(x-3)} = \lim_{x \to 3} \frac{3-x}{4(1+x)(x-3)}$$

$$= \lim_{x \to 3} \frac{-1}{4(1+x)} = -\frac{1}{16}.$$

Continued...

4. (8 points) Formulate and prove the Intermediate Value Theorem for functions f continuous on a closed interval $[a, b]$.



Suppose that f is continuous on $[a,b]$, and $f(a) < c < f(b)$ or $f(b) < c < f(a)$.
Then there exists $x_0 \in (a,b)$ s.t. $f(x_0) = c$.

Assume $f(a) < c < f(b)$, the other case follows similarly.

Proof: Let $I_1 = [a,b]$, $a_1 = a$, $b_1 = b$.

Define a_n, b_n inductively by

$$a_{k+1} = \begin{cases} a_k, & f\left(\frac{a_k+b_k}{2}\right) > c \\ \frac{a_k+b_k}{2}, & f\left(\frac{a_k+b_k}{2}\right) \le c \end{cases} \Rightarrow f(a_{k+1}) \le c$$

$$b_{k+1} = \begin{cases} \frac{a_k+b_k}{2}, & f\left(\frac{a_k+b_k}{2}\right) > c \\ b_k, & f\left(\frac{a_k+b_k}{2}\right) \le c. \end{cases} \Rightarrow f(b_{k+1}) \ge c$$

If equality is achieved in $f\left(\frac{a_k+b_k}{2}\right) = c$ for any k, we are done.

otherwise we have $b_{k+1} - a_{k+1} = \frac{b-a}{2^k} \to 0$

$I_k = [a_k, b_k]$ - nested

By the nested Intervals Principle $\exists x_0 \in \bigcap_{k=1}^{\infty} I_k$,

$a_k \le x_0 \le b_k \Rightarrow |x_0 - a_k|, |b_k - x_0| \le |b_k - a_k| \to 0$

$\Rightarrow a_k \to x_0, \; b_k \to x_0$

By Sequential definition of limit,

$f(a_k), f(b_k) \to f(x_0)$.

Continued...

Since $f(a_k) \le c \le f(b_k)$, $f(x_0) \le c$, $f(x_0) \ge c$ $\Rightarrow f(x_0) = c$.

5. (6 points) Prove that the function $f(x) = x \sin \frac{1}{x}$ is uniformly continuous on $(0, 1)$.

Set $g(x) = \begin{cases} x \sin \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$

Since $\lim\limits_{x \to 0+} x \sin \frac{1}{x} = 0$ by squeeze princ.

$\left(-x \leq x \sin \frac{1}{x} \leq x, \quad x > 0 \right)$

$g(x)$ is continuous on $[0, 1]$

(continuity for $x \neq 0$ follows from the continuity of elementary functions.)

$\Rightarrow g(x)$ is uniformly continuous on $[0, 1]$

$\Rightarrow g(x) = f(x)$ is unif. cont. on $(0, 1)$.

6. (8 points) (a) If f is uniformly continuous on an interval (a, b) and $x_n \in (a, b)$ is Cauchy, prove that $f(x_n)$ is Cauchy.

$\forall \varepsilon > 0 \quad \exists \delta > 0 \; : \; \forall x_1, x_2 \in (a, b),$

$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$

$\forall \delta > 0 \quad \exists N \in \mathbb{N} : \quad n, m > N$

$\Rightarrow |x_n - x_m| < \delta$

Thus, $\forall \varepsilon > 0$ take $N = N_\delta$ where $\delta = \delta_\varepsilon$

$\Rightarrow |f(x_n) - f(x_m)| < \varepsilon.$

(b) Can the same conclusion be made if one only requires f to be continuous on (a, b)?

No. Counterexample: $f(x) = \frac{1}{x}$

is continuous on $(0, 2)$

$x_n = \frac{1}{n} \in (0, 2), \quad n \in \mathbb{N}.$

However $f(x_n) = n \to \infty, \quad n \to \infty$

so it is not Cauchy.

7. (6 points) Let $f(x) = \sin x$, $\frac{\pi}{2} \le x \le \pi$. Use the theorem about the derivative of the inverse function to find the derivative $(f^{-1})'$. What is the domain of f^{-1}? What is the domain of $(f^{-1})'$?

Let $g = f^{-1}$.

$f(g(y)) = y$

$f'(g(y))\,g'(y) = 1$

$$g'(y) = \frac{1}{f'(g(y))}$$

$f'(x) = \cos x$

$$\Rightarrow g'(y) = \frac{1}{\cos x} = \frac{1}{-\sqrt{1-\sin^2 x}}$$

$$= \frac{1}{-\sqrt{1-y^2}}$$

$\left(\frac{\pi}{2} \le x \le \pi\right)$
$\Rightarrow \cos x < 0$

$\sin x$ is decreasing on $\left[\frac{\pi}{2}, \pi\right)$

$\Rightarrow$ range $(\sin x) = [0, 1]$

$\Rightarrow$ domain $(f^{-1}) = [0, 1]$.

$f'(x) = 0$ at $x = \frac{\pi}{2}$

$\Rightarrow g'(y)$ is undefined at $y = 1$.

domain $(g') = [0, 1)$.

8. (6 points) (a) Let $x > 0$. Prove that $(\sqrt[n]{x})^m = \sqrt[n]{x^m}$ for all n, m natural. You may use that $y = \sqrt[n]{x} \Leftrightarrow y^n = x$ and that $(x^n)^m = x^{nm}$ for all $n, m \in \mathbb{N}$.

Let $\sqrt[n]{x} = y$, then $y^n = x$ (def. of $\sqrt[n]{\cdot}$)

Let $\sqrt[n]{x^m} = z$ then $z^n = x^m$ (def of $\sqrt[n]{\cdot}$)

$\Rightarrow (y^n)^m = z^n$

$\Rightarrow y^{nm} = z^n$ (exp. laws for integer n, m)

$\Rightarrow (y^m)^n = z^n$

$\Rightarrow y^m = z$ since power fn. is one-to-one

$\Rightarrow (\sqrt[n]{x})^m = \sqrt[n]{x^m}$.

(b) Prove that $\lim_{x \to 0+} x^{\frac{m}{n}} = 0$, $\lim_{x \to +\infty} x^{\frac{m}{n}} = +\infty$ for all $n, m \in \mathbb{N}$. If you are using that the function $x \mapsto x^{\frac{1}{n}}$ is monotone, indicate where this is used, and justify this fact.

$\lim_{x \to 0+} x^{\frac{m}{n}} = 0$

Let $\varepsilon > 0$. Then $(x > 0)$ $|x^{\frac{m}{n}}| < \varepsilon \iff x^{\frac{m}{n}} < \varepsilon$

$\underset{\text{monotonicity of } x^m}{\iff} x^{\frac{1}{n}} < \varepsilon^{\frac{1}{m}} \underset{\text{monotonicity of } x^{\frac{1}{n}}}{\iff} x < \varepsilon^{\frac{n}{m}}$

Take $\delta = \varepsilon^{\frac{n}{m}}$, then $0 < x < \delta \Rightarrow |x^{\frac{m}{n}}| < \varepsilon$.

$\lim_{x \to +\infty} x^{\frac{m}{n}} = +\infty$

Let $A \in \mathbb{R}$ WLOG $A > 0$, (otherwise $A \leftarrow \max\{1, A\}$)

Then $x^{\frac{m}{n}} > A \underset{x^m \text{ is incr.}}{\iff} x^{\frac{1}{n}} > A^{\frac{1}{m}} \underset{x^{\frac{1}{n}} \text{ is incr}}{\iff} x > A^{\frac{n}{m}}$.

Let $B = A^{\frac{n}{m}}$, then

$\forall A > 0 \quad x > B = A^{\frac{n}{m}} \Rightarrow x^{\frac{m}{n}} > A$.

Continued...

9. (bonus: 6 points) (a) Suppose $b > 1$, $m \in \mathbb{N}$, and

$$x_n = \frac{b^n}{n^m}, \quad n \in \mathbb{N}.$$

Show that the sequence x_n is increasing for $n > N$. Hint: look at the quotient x_{n+1}/x_n.

$$\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{(n+1)^m} \cdot \frac{n^m}{b^n} = \left(\frac{n}{n+1}\right)^m b.$$

$$\lim \left(\frac{n}{n+1}\right)^m = \lim \left(1 - \frac{1}{n+1}\right)^m = 0$$

Thus, $\exists N \in \mathbb{N}: \quad \left(\frac{n}{n+1}\right)^m > \frac{1}{b}, \quad n > N$

Then $\quad \frac{x_{n+1}}{x_n} > \frac{1}{b} \cdot b = 1, \quad n > N$

Thus $\forall n \quad n > N \Rightarrow x_{n+1} > x_n \qquad (x_n > 0)$

By induction, $\forall n, \quad n' > n > N \Rightarrow x_{n'} > x_n$

$\Rightarrow x_n$ is increasing for $n > N$.

(b) Prove, using either part (a) or L'Hopital's rule that $\lim_{n\to\infty} x_n = +\infty$.

Using part (a): Assume x_n is bounded.

Then we must have $x_n \to L = \sup(x_n)$

$$\lim \frac{b^{n+1}}{n^m} = \lim \frac{(n+1)^m}{n^m} \lim \frac{b^{n+1}}{(n+1)^m}$$

$$= \lim \left(1 + \frac{1}{n}\right)^m \lim \underset{(n \leftarrow n+1)}{\frac{b^n}{n^m}} = 1 \cdot L = L$$

and $\lim \frac{b^{n+1}}{n^m} = b \lim \frac{b^n}{n^m} = bL$

Thus $\quad L = bL \Rightarrow (1-b)L = 0 \Rightarrow L = 0$

impossible since $b_n > b_{N+1} > 0$ for $n > N$,

Thus x_n is unbounded above, increasing

Continued...

$\Rightarrow \lim x_n = +\infty$

10. (bonus: 6 points) Suppose that f is defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

Show that $f'''(0)$ exists and find its value.

$$f'(x) = \frac{2}{x^3} e^{-1/x^2}, \quad x > 0$$

$$f''(x) = \left(-\frac{6}{x^4} + \frac{4}{x^6}\right) e^{-1/x^2}, \quad x > 0$$

$$f'''(x) = \left(\frac{24}{x^5} - \frac{36}{x^7} + \frac{8}{x^9}\right) e^{-1/x^2}, \quad x > 0$$

We have

$$\lim_{x \to 0+} \frac{e^{-1/x^2}}{x^P} = \lim_{t \to \infty} \frac{t^{P/2}}{e^t} = 0$$

$$\begin{pmatrix} t = 1/x^2 \\ x^P = t^{-P/2} \end{pmatrix}$$

by applying L'Hopital

or let $n \le t < n+1$, $m-1 \le P/2 < m$

$$\frac{e^t}{t^{P/2}} \ge \frac{e^n}{(n+1)^m} \ge \underbrace{\left(\frac{n}{n+1}\right)^m}_{\ge \frac{1}{2}m} \underbrace{\frac{e^n}{n^m}}_{\to \infty} \xrightarrow[n \to \infty]{} \infty$$

Thus,

$$\left.\begin{array}{l} f'_{right}(0) = \lim_{x \to 0+} \frac{e^{-1/x^2}}{x} = 0 \\ f'_{left}(0) = 0 \end{array}\right\} \Rightarrow f'(0) = 0$$

$$\left.\begin{array}{l} f''_{right}(0) = \lim_{x \to 0+} \frac{2e^{-1/x^2}}{x^4} = 0 \\ f''_{left}(0) = 0 \end{array}\right\} \Rightarrow f''(0) = 0$$

$$\left.\begin{array}{l} f'''_{right}(0) = \lim_{x \to 0+} \left(-\frac{6}{x^5} + \frac{4}{x^7}\right) e^{-1/x^2} = 0 \\ f'''_{left}(0) = 0 \end{array}\right\} f'''(0) = 0.$$

(Alternatively take $\lim_{x \to 0} f'''(x) = f'''(0)$.)

The end.