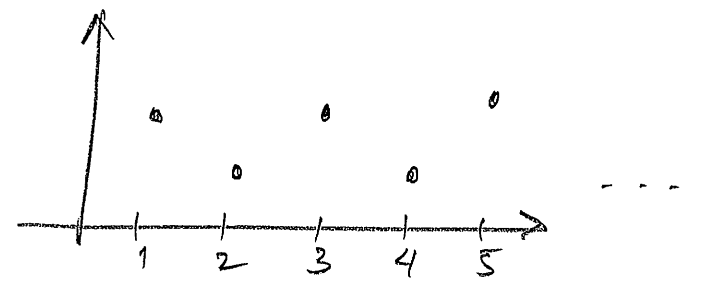
Limits of sequences: Summary

$x_n, y_n, a_n, b_n \in \mathbb{R}$ – sequences; $n \in \mathbb{N}$

Graph of a sequence:



(function, defined for $n \in \mathbb{N}$)

Def $\lim_{n\to\infty} x_n = L$

$\overset{def}{\iff}$ $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; |x_n - L| < \varepsilon$

$\lim_{n\to\infty} x_n = +\infty$

$\overset{def}{\iff}$ $\forall B \in \mathbb{R} \; \exists N \in \mathbb{N} \; \forall n > N \; x_n > B$

$\lim x_n = -\infty$

$\overset{def}{\iff}$ $\forall A \in \mathbb{R} \; \exists N \in \mathbb{N} \; \forall n > N \; x_n < A$

Notation: $\lim_{n\to\infty} x_n$ abbrev. as $\lim x_n$ or $\lim_n x_n$.

(no other limit pts. for $\mathbb{N}$ besides "∞")

Also $x_n \longrightarrow L$, $n \to \infty$

or $x_n \xrightarrow[n\to\infty]{} L$.

Basic limits

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} \pm n = \pm\infty$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Limit theorems: (Assume $\begin{cases}\lim x_n \\ \lim y_n\end{cases}$ exist.)

$$\left.\begin{aligned} \lim x_n \pm y_n &= \lim x_n \pm \lim y_n \\ \lim c x_n &= c \lim x_n \\ \lim x_n y_n &= \lim x_n \lim y_n \\ \lim y_n \neq 0 \Rightarrow \lim \frac{x_n}{y_n} &= \frac{\lim x_n}{\lim y_n} \end{aligned}\right] \text{Arithmetic operations}$$

If $f(x)$ continuous (n-independent!!)

$$\left.\lim f(x_n) = f(\lim x_n)\right] \text{Composit with cont. fn.}$$

$$\left.x_n \leq y_n \Rightarrow \lim x_n \leq \lim y_n\right] \text{Ineq.}$$

Squeeze Principle:

If $\lim x_n = \lim y_n = L$ and $x_n \leq z_n \leq y_n$ then $\lim z_n$ exist an is $= L$.

PARTICULAR CASE! $\boxed{\text{If } z_n \geq 0,\ z_n \leq y_n,\ y_n \to 0 \text{ then } z_n = c}$

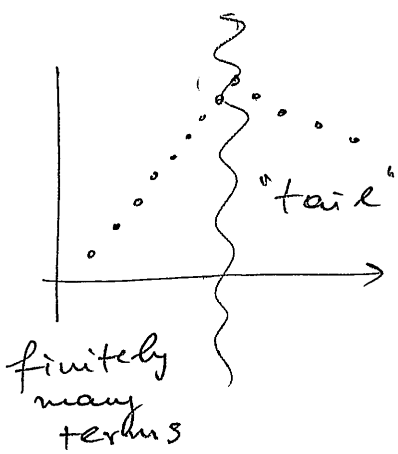
Tail principle

$(x_n)_{n\in\mathbb{N}}$ – sequence $\quad y_n = x_n, \quad n \geq n_0$

– restricted sequence, starting with $n = n_0$.

$$\boxed{\text{If } \lim y_n = L \text{ then } \lim x_n = L.}$$

Ex:
$$x_n = \begin{cases} n, & n \leq 9 \\ \frac{10}{n}, & n \geq 10 \end{cases}$$



$$y_n = \frac{10}{n}, \quad n \geq 10$$

$$\lim y_n = 10 \lim \frac{1}{n} = 0$$

$$\Rightarrow \lim x_n = 0.$$

Subsequences

$n_k \nearrow$ strictly; $\quad n_k \in \mathbb{N}$

$\Rightarrow \quad x_{n_k}$ – subsequence of (x_n)

Theorem (Subsequence Principle)

$$\left(\lim x_n = L\right) \Longleftrightarrow \left(\begin{array}{l} \forall (x_{n_k}) \text{ – subseq.} \\ \lim\limits_{k\to\infty} x_{n_k} = L. \end{array}\right)$$

Special limits

If $a > 1$

(a) $\lim_{n \to \infty} a^n = \infty$

(b) $\lim_{n \to \infty} \frac{a^n}{n} = \infty$

If $-1 < a < 1$

(c) $\lim_{n \to \infty} a^n = 0$

(d) $\lim_{n \to \infty} n a^n = 0$

Bernoulli inequality if $a > 0$

$$(1+a)^n \geq 1 + na + \frac{n(n-1)}{2} a^2.$$

Bernoulli's inequality implies (a), (b)

Proofs of (c), (d) follow by ε-N technique, since

$$|a^n| = |a|^n = \frac{1}{b^n}$$

where $b = \frac{1}{|a|}$, $b > 1 \Rightarrow$ $b^n \to \infty$

$\frac{b^n}{n} \to \infty.$

Examples 1. $\varepsilon - N$ technique

$$a_n = \sqrt{\frac{2}{n-1}}$$ guess $\lim a_n = 0$

($n-1$ in the denominator...)

Take $\varepsilon > 0$.

$$\left| \sqrt{\frac{2}{n-1}} - 0 \right| < \varepsilon \iff \sqrt{\frac{2}{n-1}} < \varepsilon$$

$$\iff \frac{2}{n-1} < \varepsilon^2 \iff n-1 > \frac{2}{\varepsilon^2}$$

$$\iff n > \frac{2}{\varepsilon^2} + 1$$

Take $N > \frac{2}{\varepsilon^2} + 1$. (by Arch. Princ.)

Then $n > N \Rightarrow n > \frac{2}{\varepsilon^2} + 1 \Rightarrow |a_n - 0| < \varepsilon$.

$$\Rightarrow \lim_{n \to \infty} a_n = 0$$

2. Composition with continuous fn:

$$a_n = \left(\frac{n-1}{n} \right)^5$$

$$a_n = \left(1 - \frac{1}{n} \right)^5 ; \quad f(x) = (1-x)^5 \text{ - continuous}$$

$$\lim \frac{1}{n} = 0 \quad \text{(basic limit)}$$

$$\lim_{n \to \infty} a_n = \left(1 - \lim \frac{1}{n} \right)^5 = 1.$$

3. Squeeze Principle; Tail principle.

$$a_n = \left(\frac{2n+3}{n^2}\right)^n$$

Let $x_n = \frac{2n+3}{n^2}$, Then $x_n = \frac{n\left(2+\frac{3}{n}\right)}{n^2}$

$$= \frac{1}{n}\left(2+\frac{3}{n}\right)$$

$$\lim x_n = \lim \frac{1}{n} \lim \left(2+\frac{3}{n}\right)$$

$$= 0 \cdot \left(2 + 3 \cdot \underbrace{\lim \frac{1}{n}}_{=0}\right) = 0.$$

Thus, $\exists N: \quad n > N \Rightarrow \quad x_n < \frac{1}{2}$

Since $x_n > 0$,

$$0 \leq a_n = (x_n)^n < \left(\frac{1}{2}\right)^n, \quad n > N$$

Since $\lim \left(\frac{1}{2}\right)^n = 0$, $\lim a_n = 0$.

Let $a > 1$,

4. $a_n = \frac{\log_a n}{n}$; $\varepsilon - N$ technique

Guess $\frac{\log_a n}{n} \longrightarrow 0$ (calculator is helpful)

Take $\varepsilon > 0$

$$\left| \frac{\log_a n}{n} - 0 \right| < \varepsilon$$

$$\Longleftrightarrow \quad \log_a n < n\varepsilon$$

$$\iff \quad n < (a^{\varepsilon})^n$$

Since $\quad a^{\varepsilon} > 1, \qquad \dfrac{(a^{\varepsilon})^n}{n} \longrightarrow \infty$

$\Rightarrow \exists N \in \mathbb{N} \quad \forall n > N$

$$\frac{(a^{\varepsilon})^n}{n} > 1 \quad \Rightarrow \quad n < (a^{\varepsilon})^n.$$

5. Squeeze Principle.

$$a_n = \underbrace{\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n+1}}}_{n+1 \text{ terms}}$$

We have $\forall k \in \{1 \dots n+1\}$

$$\frac{1}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2+k}} \leq \frac{1}{\sqrt{n^2+n+1}}$$

Thus

$$\frac{n+1}{\sqrt{n^2+1}} \leq \sum_{k=1}^{n+1} \frac{1}{\sqrt{n^2+k}} < \frac{n+1}{\sqrt{n^2+n+1}}$$

Since $\displaystyle \lim \frac{n+1}{\sqrt{n^2+1}} = \lim \frac{1+\frac{1}{n}}{\sqrt{1+(\frac{1}{n})^2}} = 1$

and $\displaystyle \lim \frac{n+1}{\sqrt{n^2+n+1}} = \lim \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+(\frac{1}{n})^2}} = 1$

by Squeeze Principle $\lim a_n = 1$.