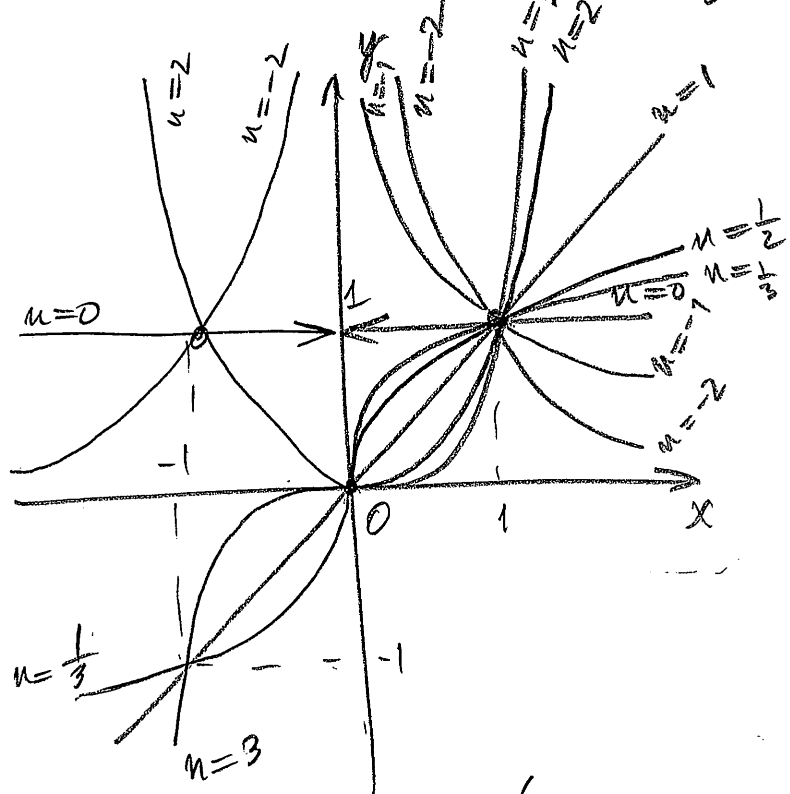
MATH 350 6/23/2015

(1)

Elementary functions — Supplement to Chapter 4

1. Power functions (Review)

1) $f(x) = x^n$, $n \in \mathbb{N}$ — by induction $(x^1 = x;\ x^{k+1} = x\,x^k)$



2) $f(x) = x^{-n}$, $n \in \mathbb{N}$

as $x^{-n} = \dfrac{1}{x^n} = \left(\dfrac{1}{x}\right)^n$,

$x \neq 0$

3) $f(x) = x^0 \equiv 1$, $x \neq 0$.

to satisfy the laws of exponents

$x^n x^m = x^{n+m}$

$(x^n)^m = x^{nm}$

for all $n, m \in \mathbb{Z}$.

4) $f(x) = x^{\frac{1}{n}}$, $n \in \mathbb{N}$ as unique soln. of $y^n = x$

(defined uniquely for $x \in [0, \infty)$, n-even

or for $x \in \mathbb{R}$, n-odd.)

5) $f(x) = x^p$, $p = \dfrac{m}{n} > 0$ — positive rational

as $x^p = \left(x^{\frac{1}{n}}\right)^m = \left(x^m\right)^{\frac{1}{n}}$ for $x \geq 0$

6) $f(x) = x^{-p}$, $p = \dfrac{m}{n} > 0$ as $x^{-p} = \dfrac{1}{x^p} = \left(\dfrac{1}{x}\right)^p$.

for $x > 0$

Laws of exponents

$$x^p x^q = x^{p+q}, \quad (x^p)^q = x^{pq}$$

hold for all $p, q \in \mathbb{Q}$, for all $x > 0$ — real.

Functions $x \mapsto x^p$ are continuous on their domains,

increasing on $(0,\infty)$ for $p > 0$
decreasing on $(0,\infty)$ for $p < 0$

$x \mapsto x^n$ are even for $n \in \mathbb{N}$ — even
$x \mapsto x^n, x^{\frac{1}{n}}$ are odd for $n \in \mathbb{N}$ — odd.

2. Exponential and logarithmic functions (New)

Let $a > 1$ — real; a^r — defined for $r \in \mathbb{Q}$ using def. of power.

1° a^r — increasing on $\mathbb{Q}$: $r < s \Rightarrow a^r < a^s$

2° $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = 1$ (used Bernoulli ineq.) $\lim\limits_{n\to\infty} a^{-\frac{1}{n}} = 1$ (take reciprocal)

3° $\lim\limits_{\mathbb{Q} \ni r \to 0} a^r = 1$, i.e. $\forall \varepsilon > 0 \ \exists \delta > 0$: $-\delta < r < \delta \Rightarrow |a^r - 1| < \varepsilon$

Indeed, take N such that

$$1 - \varepsilon < a^{-\frac{1}{N}} < a^{\frac{1}{N}} < 1 + \varepsilon, \quad \delta = \frac{1}{N}$$

$$\Rightarrow \left(-\delta < r < \delta \Rightarrow 1 - \varepsilon < a^{-\frac{1}{N}} < a^r < a^{\frac{1}{N}} < 1 + \varepsilon\right).$$

4°. Let $x \in \mathbb{R}$. Then

$$\sup_{\mathbb{Q} \ni r_1 < x} a^{r_1} = \inf_{\mathbb{Q} \ni r_2 > x} a^{r_2}$$

Proof: $r_1 < x < r_2 \Rightarrow a^{r_1} < a^{r_2}$

$$\Rightarrow \forall r_1 < x \quad a^{r_1} \leq \inf_{r_2 > x} a^{r_2}$$

$$\Rightarrow \sup_{r_1 < x} a^{r_1} \leq \inf_{r_2 > x} a^{r_2}$$

Let $s = \sup\limits_{r_1 < x} a^{r_1}$; $t = \inf\limits_{r_2 > x} a^{r_2}$.

Then $\forall r_1, r_2 \quad r_1 < x < r_2 \Rightarrow a^{r_1} \leq s \leq t \leq a^{r_2}$

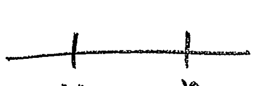
$$\Rightarrow \quad t - s \leq a^{r_2} - a^{r_1}$$
$$= a^{r_1}(a^{r_2 - r_1} - 1) \leq s(a^{r_2 - r_1} - 1)$$

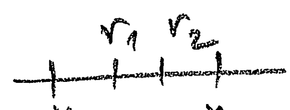
$\forall \varepsilon > 0$ take δ : $|r| < \delta \Rightarrow |a^r - 1| < \frac{\varepsilon}{s}$

$$\Rightarrow \forall \varepsilon > 0 \ \propto t - s < \varepsilon \Rightarrow t = s.$$

Def: (Exponential fn.) $a^x = \sup\limits_{r_1 < x} a^{r_1} = \inf\limits_{r_2 > x} a^{r_2}$

5°. The function $x \longmapsto a^x$ is str. increasing.



$\Rightarrow$


$\Rightarrow a^{x_1} \leq a^{r_1} < a^{r_2} \leq a^{x_2}$

6°. $$a^x = \lim_{\mathbb{Q} \ni r \to x} a^r$$

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0 : 0 < |r - x| < \delta \Rightarrow |a^r - a^x| < \varepsilon$.

Since $a^x = \sup_{r_1 < x} a^{r_1} = \inf_{r_2 > x} a^{r_2}$

$\forall \varepsilon > 0 \quad \exists r_1, r_2 : \quad r_1 < x < r_2$

$$a^x - \varepsilon < a^{r_1} < a^x < a^{r_2} < a^x + \varepsilon$$

$\Rightarrow$ for $0 < |r - x| < \delta$, $\delta = \min\{r_2 - x, x - r_1\}$

$$a^x - \varepsilon < a^r < a^x + \varepsilon$$

7°. The exponential laws:

$$a^{-x} = \frac{1}{a^x}$$

$$a^x a^y = a^{x+y}$$

$$(a^x)^y = a^{xy}$$

carry over to the case $x, y \in \mathbb{R}$.

Ex: if $x \in \mathbb{R}$

$$a^{-x} = \lim_{\mathbb{Q} \ni r \to -x} a^r = \lim_{\mathbb{Q} \ni q \to x} a^{-q} = \lim_{\mathbb{Q} \ni q \to x} \frac{1}{a^q}$$

$$= \frac{1}{\lim_{q \to x} a^q} = \frac{1}{a^x}$$

8° $\lim\limits_{x\to 0} a^x = 1$

Indeed, $\forall \varepsilon\ \exists N : n > N \Rightarrow 1-\varepsilon < a^{-\frac{1}{n}} < a^{\frac{1}{n}} < 1+\varepsilon.$

Take any $n > N$, for instance $n = N+1$.

Then $-\frac{1}{n} < x < \frac{1}{n} \Rightarrow 1-\varepsilon < a^{-\frac{1}{n}} < a^x < a^{\frac{1}{n}} < 1+\varepsilon$

Thus, $\delta = \frac{1}{N+1}$ works in the def. of limit.

$\Rightarrow \lim\limits_{x\to 0} a^x = 1.$

9° The function $x \longmapsto a^x$ is continuous on $\mathbb{R}$.

Indeed $\lim\limits_{x\to x_0} a^x - a^{x_0} = \lim\limits_{x\to x_0} a^{x_0}(a^{x-x_0}-1)$

$= a^{x_0} \lim\limits_{t\to 0} (a^t - 1) = a^{x_0}.$

10° $\forall x \in \mathbb{R}$ $a^x > 0$ (by construction)

$\lim\limits_{x\to\infty} a^x = \infty$; $\lim\limits_{x\to-\infty} a^x = 0.$

Indeed, $\lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}}} a^n = \infty$ (used Bernoulli!)

$\forall A \in \mathbb{R}\ \exists N \in \mathbb{N} : a^N > A$

$\Rightarrow \forall x > N\ a^x > a^N > A$

If $b = \frac{1}{a}$ then $0 < b < 1 \Rightarrow \lim b^n = 0$

$\forall \varepsilon > 0\ \exists N \in \mathbb{N} : b^N < \varepsilon \Rightarrow \forall x < -N$

$$a^x < a^{-N} = \left(\frac{1}{a}\right)^N = b^N < \varepsilon.$$

11°   If $f(x) = a^x$ then $f(\mathbb{R}) = (0, +\infty)$.

Indeed $\forall y \in (0, +\infty)$

$\exists\ x_1, x_2 \quad x_1 < x_2, \quad f(x_1) < y < f(x_2)$

By I.V.T $\quad \exists\ x_0 \in \mathbb{R}, \quad a^{x_0} = y.$

Since a^x is str. increasing, x_0 is unique.

<u>Def</u>: (Logarithmic function.) Let $f: \mathbb{R} \to (0, \infty)$, $f(x) = a^x$ be the exp. fn. with base $a > 1$. The inverse fn.

$$f^{-1}: (0, \infty) \to \mathbb{R}$$

is called the logarithmic fn with base a, denoted $f^{-1}(x) = \log_a x$.

<u>Rmk</u>: 1) $\log_a x$ is increasing and continuous on $(0, \infty)$. by Thms about inverse fns

2) The same construction can be done when $0 < a < 1$; Then the resulting exp. fn. is decreasing. In that case also $a^x = \frac{1}{b^x}$, where $b = \frac{1}{a} > 1$.

Laws of logarithms:

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a(x^y) = y \log_a x$$

$$\log_a a = 1$$

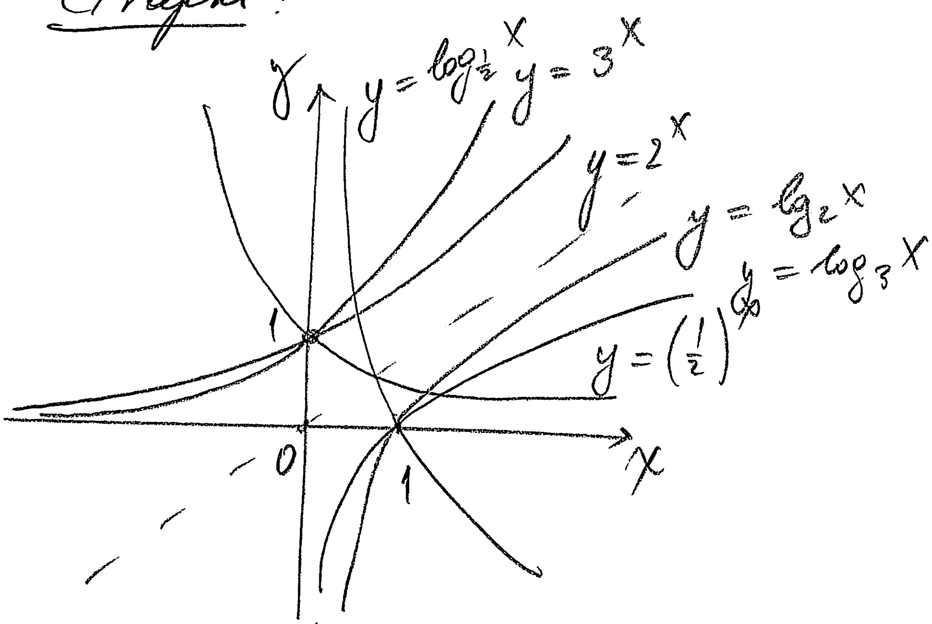
follow from the corresp. laws of exponents.

Also, if $b > 1$, then $b = a^{\log_a b}$

$$\Rightarrow \quad a^{\log_a b \cdot \log_b x} = a^{\log_a x}$$

$$\Rightarrow \log_b x = \frac{\log_a x}{\log_a b}.$$

Graphs:



12° Power function $f(x) = x^p$ with $p \in \mathbb{R}$

Def: $x^p := a^{p \log_a x}$, $x > 0$, for any $a > 1$

This is independent of the value of a, since if $b > 1$, $b^{p \log_b x} = a^{p \log_a b \log_b x} = a^{p \log_a x}$.

When $p \in \mathbb{Q}$, $a^{p \log_a x} = \left(a^{\log_a x}\right)^p = x^p$,

in accordance with the old definition.

Since $\exp_a$ and $\log_a$ functions are continuous, $f(x) = x^p$ is continuous on $(0, \infty)$

For $p > 0$ we have

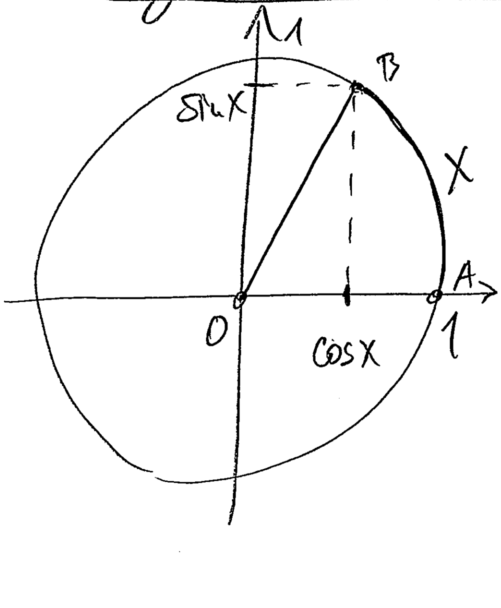
$$\lim_{x \to 0+} x^p = \lim_{x \to 0+} a^{p \log_a x}$$

$$= \lim_{y \to -\infty} a^{py} = 0,$$

so x^p, $p>0$ can be extended to $[0, +\infty)$ by setting $0^p = 0$

(Notice that 0^0 remains undefined.)

3. Trigonometric functions



1° Geometric definitions:

Via coordinate system and the unit circle.

x – length of the arc $\overline{AB}$ (radian measure of the angle.)

2° Based on the definitions,

$$-1 \le \sin x \le 1 \qquad \sin(-x) = -\sin x \text{ (odd)}$$

$$-1 \le \cos x \le 1 \qquad \cos(-x) = \cos x \text{ (even)}$$

$$\sin^2 x + \cos^2 x = 1 \quad \text{(Pythagorean thm)}$$

$$\left.\begin{aligned}
&\sin x = 0 \text{ for } x = \pi n,\\
&\cos x = 0 \text{ for } x = \pi(n+\tfrac{1}{2}),\\
&\sin x = 1 \text{ for } x = \pi(2n+\tfrac{1}{2})\\
&\sin x = -1 \text{ for } x = \pi(2n+\tfrac{3}{2})\\
&\cos x = 1 \text{ for } x = 2\pi n\\
&\cos x = -1 \text{ for } x = \pi(2n+1)
\end{aligned}\right\} n \in \mathbb{Z},$$

Also $\tan x = \dfrac{\sin x}{\cos x}$; $\cot x = \dfrac{\cos x}{\sin x}$

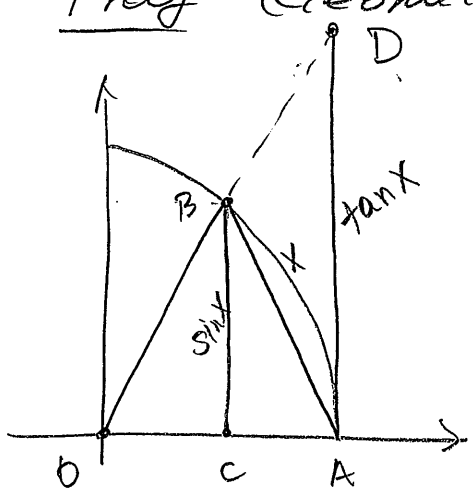
$\sec x = \dfrac{1}{\cos x}$, $\csc x = \dfrac{1}{\sin x}$ ...

3° <u>Important inequality</u>

$$\cos x \le \frac{\sin x}{x} \le 1, \quad 0 < |x| < \frac{\pi}{2}.$$

<u>Proof</u> (Geometric) Enough to show for $0 < x < \frac{\pi}{2}$

Since $\cos x$, $\dfrac{\sin x}{x}$ - even



$\overset{\frown}{AB} = x$ (arc length)

$$\sin x = \overline{BC} < \underset{\text{(chord)}}{\overline{AB}} < \overset{\frown}{AB} = x$$

↑ pyth. Thm ↖ straight line shortest path

Further $\overline{AD} = \tan x$, $\overline{OA} = 1$

$$\text{area}(\triangle OAD) = \frac{1}{2}\tan x$$

$$\text{area}(\measuredangle OAB) = \frac{1}{2} x$$

(circ. sector)

$$\Rightarrow \frac{1}{2} x < \frac{1}{2}\tan x \Rightarrow x < \frac{\sin x}{\cos x}$$

$$\Rightarrow \cos x < \frac{\sin x}{x} \quad \left(0 < x < \frac{\pi}{2}\right)$$

4° Important limits:

$$\lim_{x \to 0} \sin x = 0$$

since $\left|\frac{\sin x}{x}\right| \le 1 \Rightarrow |\sin x| \le |x|$

$\Rightarrow$ use Squeeze Principle.

$$\lim_{x \to 0} \cos x = 1$$

since $\lim_{x \to 0} (1 - \cos x) = \lim_{x \to 0} 2 \sin^2 \frac{x}{2}$

(Here used $\frac{1 - \cos(2x)}{2} = \sin^2 x$)

$$= 2 \left(\lim_{x \to 0} \sin \frac{x}{2}\right)^2$$

$$= 2 \left(\lim_{y \to 0} \sin y\right)^2 = 0.$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{by Sq. Princ.}$$

since $\cos x \le \frac{\sin x}{x} \le 1$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \frac{1}{2}\left(\lim_{y \to 0} \frac{\sin y}{y}\right)^2 = \frac{1}{2}.$$

5° Continuity of trig functions.

Trig. identies for sums of angles

$$\sin(x+y) = \sin x \cos x + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Based on these,

$$\lim_{h\to 0}\left(\sin(x+h) - \sin(x)\right) = \lim_{h\to 0} \sin x\,(\cos h - 1) + \lim_{h\to 0} \cos x \sin h$$
$$= \sin x \lim_{h\to 0}(\cos(h) - 1) + \cos x \lim_{h\to 0} \sin(h)$$
$$= \sin x \cdot 0 + \cos x \cdot 0 = 0.$$

(Here, used $\lim_{h\to 0}(\cos(h)-1) = \lim_{h\to 0} h^2 \lim_{h\to 0} \frac{\cos(h)-1}{h^2} = 0 \cdot \frac{1}{2} = 0$)

$$\lim_{h\to 0}\left(\cos(x+h) - \cos(x)\right) = \lim_{h\to 0} \cos x\,(\cos h - 1) - \lim_{h\to 0} \sin x \cdot \sin(h)$$
$$= \cos x \lim_{h\to 0}(\cos(h) - 1) - \sin x \lim_{h\to 0} \sin(h)$$
$$= \cos x \cdot 0 - \sin x \cdot 0 = 0.$$

$\Rightarrow$ $\sin x$, $\cos x$ are continuous on $\mathbb{R}$.

6° Derivatives of trig functions

Based on the calculations above,

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \sin x \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = \cos x$$

since $\lim_{h\to 0} \frac{\cos(h)-1}{h} = \lim_{h\to 0} h \lim_{h\to 0} \frac{\cos(h)-1}{h^2} = 0\,\frac{1}{2} = 0.$

$$\lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} = \cos x \lim_{h\to 0}\frac{\cos(h)-1}{h}$$

$$- \sin x \lim_{h\to 0}\frac{\sin h}{h} = -\sin x.$$

Thus, $\sin(x)$, $\cos(x)$ are differentiable on $\mathbb{R}$, and

$$(\sin x)' = \cos x\,; \quad (\cos x)' = -\sin(x).$$

Using quotient rules,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

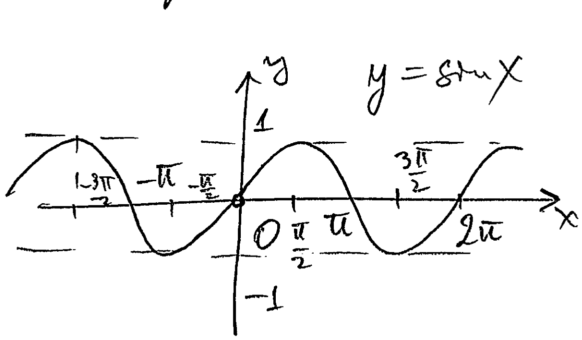
$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = -\frac{1}{\cos^2 x}(-\sin x) = \frac{\sin x}{\cos^2 x}$$

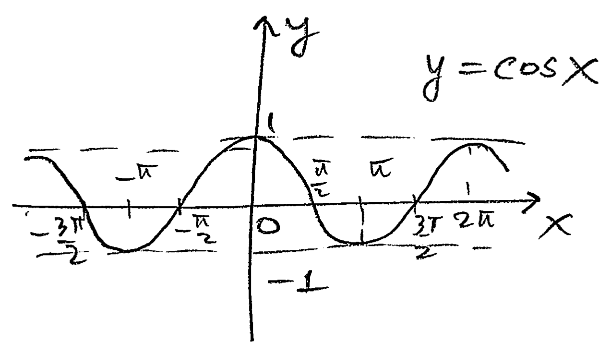
$$= \tan x \sec x$$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = -\frac{1}{\sin^2 x}(\cos x) = -\frac{\cos x}{\sin^2 x}$$

$$= -\cot x \csc x.$$

Graphs:





4. Derivatives of exponential and logarithmic functions.

1°. The number "e":

Recall the definition:

Def: $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ – The Euler number "e"
– the base of the natural logarithm.

Recall: The limit exists, since the sequence $\left(1+\frac{1}{n}\right)^{n+1}$ is positive and decreasing;

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \frac{\lim\left(1+\frac{1}{n}\right)^{n+1}}{\lim\left(1+\frac{1}{n}\right)} = \lim\left(1+\frac{1}{n}\right)^{n+1}.$$

Since $\left(1+\frac{1}{n}\right)^n$ is increasing,

$$\forall n \in \mathbb{N} \qquad \left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

so for instance $(n=100)$ $\quad 2.7048 < e < 2.7319$

(not the most efficient way to compute approximations of e, but it works)

with higher accuracy, $e = 2.718281828459\ldots$

2°. $e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$.

Indeed, for $x > 1$ let $n \in \mathbb{N}$ be such that $n \le x < n+1$

($n = \max\{x' \in \mathbb{N} : x' \le x\}$ – the integer part of x)

$$\forall \varepsilon > 0 \;\; e - \varepsilon < \left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1} < e + \varepsilon$$

for large enough n, since

$$\lim\left(1+\frac{1}{n+1}\right)^n = \lim\left(1+\frac{1}{n}\right)^{n+1} = e,$$

3° Let $a>1$; $f(x)=a^x$ – exp. fn.

Then

$$\lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

$$\underset{\uparrow}{=} \lim_{t\to 0}\frac{t}{\log_a(1+t)} = \frac{1}{\lim\limits_{t\to 0}\frac{\log_a(1+t)}{t}}.$$

$$\begin{pmatrix} t = a^h - 1 \\ h = \log_a(1+t) \end{pmatrix}$$

Now,

$$\lim_{t\to 0}\frac{\log_a(1+t)}{t} = \lim_{t\to 0}\log_a(1+t)^{\frac{1}{t}}$$

$$\underset{\nearrow}{=} \lim_{z\to+\infty}\log_a\left(1+\frac{1}{z}\right)^z = \log_a \lim_{z\to+\infty}\left(1+\frac{1}{z}\right)^z$$

$$\left(z=\frac{1}{t}\right) \qquad = \log_a e.$$

Thus

$$(a^x)' = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \frac{a^x}{\log_a e} = (\ln a)a^x.$$

<u>Notation</u>: $\ln = \log_e$; also denoted by "$\log$"

$\exp(x) = e^x$. $\exp_a(x) = a^x$.

In particular, if $a=e$, then

$$(e^x)' = e^x.$$

For the derivative of $\log$, use the inverse function:

$$e^{\ln x} = x \qquad\qquad a^{\log_a x} = x$$

$$e^{\ln x}(\ln x)' = 1 \qquad\qquad (\ln a)\cdot a^{\log_a x}(\log_a x)' = 1$$

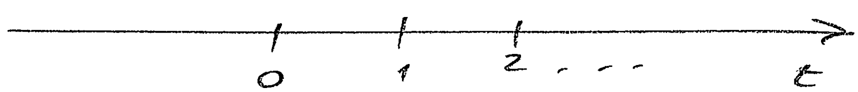
$$x\,(\ln x)' = 1 \qquad\qquad x \ln a\,(\log_a x)' = 1$$

$$\ln x = \frac{1}{x}. \qquad\qquad (\log_a x)' = \frac{1}{x \ln a}$$

Proved that $\exp_a$; $\log_a$ are differentiable on $\mathbb{R}$ and $(0,\infty)$, respectively.

4° The exponential function and the continuous growth-decay processes.

$\mathbb{R}$ represents the time-axis:



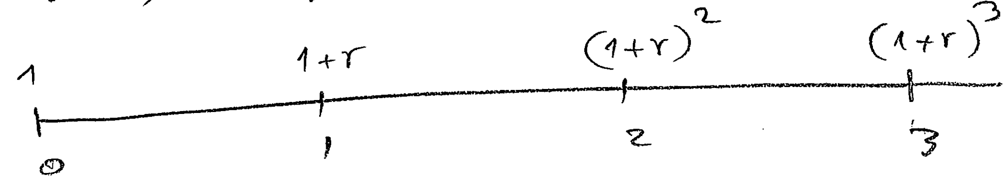
If time is discrete, a geometric sequence

$$(1+r)^n,\ n = 0, 1, 2 \ldots \qquad (r > -1)$$

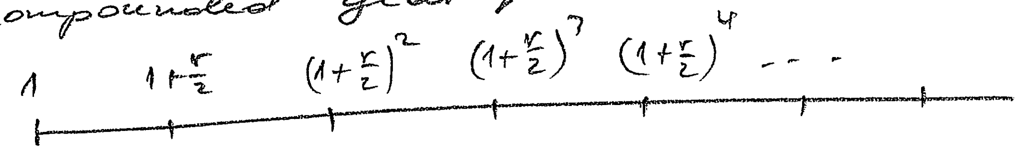
represents a process of growth/decay of an initial amount 1 at a constant per-capita rate r.

Example: Compound interest

$A(0)=1$, $A(n) = (1+r)^n$, $n \in \mathbb{N}$



compounded yearly



compounded every six months

compounded n times a year

In the limit $n \to \infty$ obtain a cont. time process of increase/decrease at constant per-capita rate r;

Amount after 1 time period:

$$A(1) = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right)^r$$

$$= \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^r = e^r$$

The amount at time $t = \frac{k}{n}$ is

$$A(t) = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^k = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

This leads to another possible definition of the exp. function:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Clearly, positive; increasing with x;
If one is able to prove continuity, and that $\sup e^x = \infty$, $\inf e^x = 0$;
then I.V.T. can be used to define $\ln x$; then $a^x := e^{x\ln a}$

$$x^p = e^{p\ln x}$$

— alternative definitions of the general exponential and power functions.