Completeness axiom and the Archimedian principle.

Axiom C: For every nonempty $A, B \subseteq \mathbb{R}$ such that $\forall x \in A \ \forall y \in B \ \ x \le y$ there exist $c \in \mathbb{R}$: $\forall x \in A \ \forall y \in B \ \ x \le c \le y$

Def: $A \subseteq \mathbb{R}$; then $y \in \mathbb{R}$ is an <u>upper bd.</u> of A if $\forall x \in A \ \ x \le y$.

$y = \max A$ if y is an upper bound and $y \in A$.

(maximum element of A.)

Likewise, $B \subseteq \mathbb{R}$, then $x \in \mathbb{R}$ is a <u>lower bound</u> of B if $\forall y \in B \ \ x \le y$.

$x = \min B$, if x is a lower bound, and $x \in B$.

Def: A is <u>bounded above</u> if $\exists y \in \mathbb{R}$ - upper bound.

B is <u>bounded below</u> if $\exists x \in \mathbb{R}$ - lower bound.

A or B is <u>bounded</u> if it is bounded above and below.

Ex: $[0, \infty)$ is unbounded above.

It is bounded below by 0, or any negative number.

$0 = \min[0,\infty)$.

$[1,2)$ is bounded above and below

$1 = \min[1,2)$, there is no max.

Rmk: If min / max exist they are necessarily unique (prove!)

Lemma (Archimedean principle)

$\forall a\in\mathbb{R}\ \exists N\in\mathbb{N}: N>a$.

$\iff$ $\mathbb{N}$ is unbounded above.

$\iff \forall a \in\mathbb{R},\ A=\{x\in\mathbb{N}: x\le a\}\neq\mathbb{N}$.

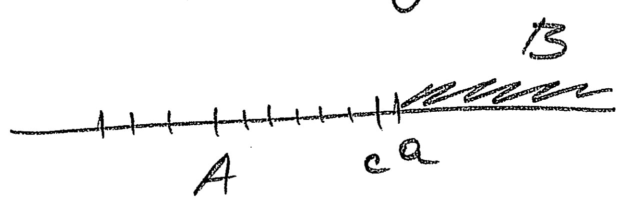
Proof: Let $a\in\mathbb{R}$. WTS $A=\{x\in\mathbb{N}: x\le a\}\neq\mathbb{N}$.

Consider $B=\{y\in\mathbb{R}: \forall x\in A\ x\le y\}$

(B is nonempty since $a\in B$.) (set of upper bounds of A.)

By Axiom C, $\exists c\in\mathbb{R}$

$\forall x\in A\ \forall y\in B\ x\le c\le y$.



Then $c=\min B$.

$c-1<c$, so $c-1\notin B$.

$c-1$ is not an upper bound of A

$\Rightarrow \exists\, n \in A,\ n > c-1$, or

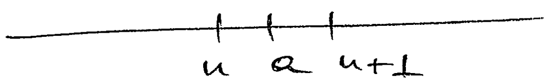
$c-1 < n \leq c$ (since c is an upper bd.).

Then $n+1 > c$ so

$n+1 \notin A$ (since all elements in A are $\leq c$)

Thus, $n+1 > a$, $n \leq a$, so

$a-1 < n \leq a$ We can then take $N = n+1$.



$\square$

Corollary: Every bounded subset of $\mathbb{N}$ has a maximal element.

Same for $\mathbb{Z}$.

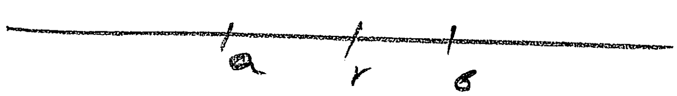
Corollary: $\forall a \in \mathbb{R}$ $\exists n \in \mathbb{N}$: $a-1 < n \leq a$.

Corollary: If $b \in \mathbb{R}$ (big) and $a \in \mathbb{R}$ (small) then $\exists n \in \mathbb{N}$: $na > b$.

Proof: The set $\{n : n \leq \frac{b}{a}\}$ is unbounded.

Application: Density of rational/irrational numbers.

1°. Between any two real numbers there is a rational number.



Clear if $a, b \in \mathbb{Q}$ ($r = \frac{a+b}{2}$) If not, take $\frac{1}{n} < b-a$;

$\frac{m}{n} < b; \quad \frac{m+1}{n} \geq b$

Take m: $\frac{m}{n} \leq b, \quad \frac{m+1}{n} \geq b$

$a \quad \frac{m}{n} \quad b \quad \frac{m+1}{n}$

Then $\frac{m}{n} - a = \frac{m}{n} - b + (b-a)$

$> \frac{m}{n} - b + \frac{1}{n} = \frac{m+1}{n} - b > 0$

So $a < \frac{m}{n} \leq b$.

2° Between any two real numbers there is an irrational number.

$\frac{m}{n}\sqrt{2}$ is irrational;

Take n: $\frac{\sqrt{2}}{n} < b - a$.

Then $\exists m$: $a < \frac{m\sqrt{2}}{n} \leq b$

(what if b is irrational, how about $\frac{m\sqrt{2}}{n} = b$?)

Could take $\frac{1}{n} < \frac{b-a}{2}$, then $a < b - \frac{1}{n} < b$, repeat with $\frac{1}{n} < \frac{b-a}{2}$ ...

Example: Discontinuity of the Dirichlet function.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then $\forall a \in \mathbb{R}$ $f(x)$ is discontinuous at a.

Proof: Take $a \in \mathbb{R}$; Let $L \in \mathbb{R}$.

$\varepsilon = \max\{|L|, |L-1|\}$.

Then $\forall \delta \ \exists x_\delta'$: $|x_\delta - L| = |L-1|$

$\exists x_\delta''$: $|x_\delta - L| = |L|$

So $\forall\, \delta > 0 \quad \exists\, x_\delta : \quad |x_\delta - a| < \delta :$

$$|f(x) - L| \geq \varepsilon$$

$(=)$

$\boxed{\checkmark}$

Problem: Prove that any interval contains ∞-many irrational numbers, ∞ many rational numbers.

Case 1 : $a, b \in \mathbb{Q}$. Take $a_0 = a$, $a_1 = \frac{a+b}{2}$,

$$a_{n+1} = \frac{a_n + a_{n-1}}{2}.$$

$\vdots$

Then $a < a_n < b$.

Rmk: Proved by induction, using

$$a = \frac{a}{2} + \frac{a}{2} < \frac{a+b}{2} < \frac{b}{2} + \frac{b}{2} = b.$$

Case 2 : $a, b \in \mathbb{R}$, rational pts.

take $\frac{1}{n} < \frac{b-a}{2}$; $\frac{m}{n} < b$ closest to b;

then $a < \frac{m-1}{n} < \frac{m}{n} < b$

Repeat the process of Case 1 for $a_0 = \frac{m-1}{n}$, $a_1 = \frac{m}{n}$.

Case 3: $a, b \in \mathbb{R}$, irrational pts.

take $a_0 \in \mathbb{Q}$, $a < a_0 < b$.

Take $a_1 \in \mathbb{R} \setminus \mathbb{Q}$: $a_0 < a_1 < b$.

Repeat the above process with a_0, a_1. all a_n are irrational.