“Web Problems” for Homework 8

1. Give a direct proof that $(0.4)^x$ is unbounded above, without refering to the facts about the logarithmic function.

2. Prove that $10^{-|x|}$ is bounded. Find the supremum and infimum on $\mathbb{R}$ (with proofs).

3. (a) Show that $f(x) = 2^{1/x}$ is decreasing on any interval which does not contain zero.
   (b) Show that $\inf_{(-\infty,0)} f = 0$, $\sup_{(-\infty,0)} f = 1$, $\inf_{(0,\infty)} f = 1$, $\sup_{(0,\infty)} f = +\infty$.
   (c) Prove that the range of f is $(0, \infty) \setminus \{1\}$.
   (d) Find the inverse function f^{-1}.
   (e) Find the range of f^{-1}.
   (f) Find the derivative $(f^{-1})'$.

4. (a) Prove that if $\max_X f$ exists then
$$\sup_X f = \max_X f.$$
   (b) Prove that if $\min_X f$ exists then
$$\inf_X f = \min_X f.$$

5. Prove that
$$\sup_X (f + g) \leq \sup_X f + \sup_X g,$$
and
$$\inf_X (f + g) \geq \inf_X f + \inf_X g.$$

6. Prove that
$$\sup_X (-f) = -\inf_X f,$$
and
$$\inf_X (-f) = -\sup_X f.$$

7. If f does not change sign on f, prove that
$$\sup_X \frac{1}{f} = \frac{1}{\inf_X f}, \quad \text{and} \quad \inf_X \frac{1}{f} = \frac{1}{\sup_X f}.$$

8. (a) Prove that if f is decreasing on $[a, b]$ and continuous, then f^{-1} exists and is continuous on $[f(b), f(a)]$.

   (b) Prove that if f is increasing on $[a, b]$ and f^{-1} exists then f^{-1} is increasing on $[f(a), f(b)]$.

   (c) Prove that if f is decreasing on $[a, b]$ and f^{-1} exists then f^{-1} is decreasing on $[f(b), f(a)]$.

9. Find the limits of sequences:

   (a) $\left(\dfrac{n+10}{2n-1}\right)^n$

   (b) $\left(1+\dfrac{4}{n}\right)^n$

   (c) $\left(\dfrac{n+1}{n-1}\right)^n$

   (d) $\dfrac{2n(-1)^n+3}{4-\sqrt{n^2+5}(-1)^n}$

   (e) $n \sin \dfrac{\pi}{n}$

   (f) $n \cos \dfrac{\pi}{n}$.

10. Find the derivatives of $f(x)$ at $x = x_0$ based on the definition:

   (a) $f(x) = \sqrt{x}$, $x_0 > 0$

   (b) $f(x) = \sqrt{x+2}$, $x_0 = 1$

   (c) $f(x) = 1/(1+x)$, $x_0 = 3$

   (d) $f(x) = x^{1/3}$, $x_0 \neq 0$

   (e) $f(x) = \sin x$, $x_0 \in \mathbb{R}$

   (f) $f(x) = \cos x$, $x_0 \in \mathbb{R}$.