"Web Problems" for Homework 5

1. (After Problems 4, 6 in Section 2.5)

   (a) If $-1 < a < 1$ prove that $\lim_n n\, a^n = 0$.

   (b) If $a > 1$ prove that $\lim_n \frac{a^n}{n} = \infty$.

   (c) If $a > 1$ and $m \in \mathbb{N}$ prove that $\lim_n \frac{a^n}{n^m} = \infty$.

2. (After Problems 11-15 in Section 2.5) Use the ε-N technique or theorems about limits to find the limits of sequences, or show they do not exist:

   (a) $n(-1)^n$

   (b) $\sin \frac{\pi n}{180}$

   (c) $\sqrt{\frac{2}{n-1}}$

   (d) $\sqrt{n^2+n} - n$

   (e) $\left(\frac{n-1}{n}\right)^5$

   (f) $\frac{(-1)^n + \frac{1}{n}}{\frac{1}{n^2} - (-1)^n}$

   (g) $\frac{5^n}{n^n}$

   (h) $\left(\frac{2n+3}{n^2}\right)^n$

   (i) $\frac{\log n}{n}$

   (j) $\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n+1}}$.

3. (a) Let $a > 1$. Prove that $\lim_n \sqrt[n]{a} = 1$.

   (b) Prove that $\lim_n \sqrt[n]{n} = 1$.

   (c) Let $0 < a < 1$. Prove that $\lim_n \sqrt[n]{a} = 1$.

   Hint for (a): Let $\sqrt[n]{a} - 1 = b_n$. Then $b_n > 0$ and $a = (1 + b_n)^n \geq n b_n$ by Bernoulli's inequality.

4. Let $f : [a, b] \to \mathbb{R}$ be strictly increasing and continuous, and let $c = f(a)$, $d = f(b)$.

   (a) Show that f is one-to-one.

   (b) Use the intermediate value property to argue that $f([a, b]) = [c, d]$.

   (c) Show that the inverse function $f^{-1} : [c, d] \to [a, b]$ is continuous on $[c, d]$.

5. Given $n \in \mathbb{N}$, let $f : \mathbb{R} \to \mathbb{R}$ be the power function $f(x) = x^n$.

   (a) Show that f is strictly increasing on $[0, \infty)$.

   (b) Show that f is an even function for n even, and an odd function for n odd.

   (c) Show that f is unbounded above (i. e. the range $f(\mathbb{R})$ is an unbounded above set).

   (d) If n is odd, show that f is unbounded below.

(e) Use the above and the result of the previous problem to show that the equation $y^n = x$ has a unique solution y (denoted $y = \sqrt[n]{x}$) for every $x \in \mathbb{R}$ when n is odd, and for every $x \geq 0$ when n is even.

(f) Conclude that the function $\sqrt[n]{x}$ is continuous on $\mathbb{R}$ for n odd, and continuous on $[0, \infty)$ for n even.

6.* Let $x \geq 0$ be real and let $p = \frac{m}{n} > 0$ be rational. Define $x^p := \left(\sqrt[n]{x}\right)^m$.

(a) Show that x^p is well-defined and does not depend on the representation of p as a fraction (i. e. $\left(\sqrt[nk]{x}\right)^{mk} = \left(\sqrt[n]{x}\right)^m$ for any $k \in \mathbb{N}$).

(b) Show that $\left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$.

(c) Show that $x^p x^q = x^{p+q}$ and $(x^p)^q = x^{pq}$.

(d) Use the results of the previous problem and theorems on continuous functions to argue that $f(x) = x^p$ is continuous on $[0, \infty)$.

Remark: The equality in 6 (b) is no longer true if $x < 0$: for instance, $\left((-1)^2\right)^{\frac{1}{6}} = 1$ while $\left((-1)^{\frac{1}{6}}\right)^2$ is undefined. Moreover $\left((-1)^1\right)^{\frac{1}{3}} = -1$, so trying to define $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ and using this for $x < 0$ leads to problems.

7.* (Negative exponents)

(a) Let $x \in \mathbb{R}$, $x \neq 0$. Show that $\forall m \in \mathbb{N}$ $(x^{-1})^m = (x^m)^{-1}$.

(b) Let $x \neq 0$ be such that $\sqrt[n]{x}$ is defined. Show that $\forall n \in \mathbb{N}$ $\sqrt[n]{x^{-1}} = (\sqrt[n]{x})^{-1}$.

(c) Let $x > 0$ real, $p > 0$ rational. Show that $(x^{-1})^p = (x^p)^{-1}$.

Remark: Results of the last problem allow us to define x^{-m}, $x^{-\frac{1}{n}}$ and x^{-p} as either of the equal expressions. Using the agreement $x^0 = 1$, $x \neq 0$ one can obtain the laws of exponents in 6 (c) for all $x > 0$ and all $p, q \in \mathbb{Q}$.

8.* (a) Prove that if $p > 0$ is rational then $\lim\limits_{x \to +\infty} x^p = +\infty$, $\lim\limits_{x \to 0+} x^p = 0$.

(b) Prove that if $p < 0$ is rational then $\lim\limits_{x \to +\infty} x^p = 0$, $\lim\limits_{x \to 0+} x^p = +\infty$.

(c) Prove that if $P(x)$ is a polynomial of odd degree then $\lim\limits_{x \to \pm\infty} P(x) = \pm\infty$.

(d) Prove that if $P(x)$ is a polynomial of even degree then $\lim\limits_{x \to \pm\infty} P(x) = +\infty$.

Note: Problems marked with * are for the Practice section.