Homework Assignment 1

Quiz on Mon. Jun 1, 2015, in class.

Note: For problems with proofs try to follow formal logical notation and show all steps.

1. Use truth tables to show that for any logical statements P, Q, R one has
   (a) $\overline{\overline{P}} = P$
   (b) $\overline{(P \vee Q)} \Leftrightarrow \overline{P} \wedge \overline{Q}$
   (c) $\overline{(P \wedge Q)} \Leftrightarrow \overline{P} \vee \overline{Q}$
   (d) $(P \Rightarrow Q) \Leftrightarrow \overline{P} \vee Q$
   (e) $(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$
   (f) $(P \vee Q) \wedge R \Leftrightarrow (P \wedge R) \vee (Q \wedge R)$.

2. Which of the logical statements P, Q, R have to be true or false if we know that the statement
$$\big((\overline{\overline{P} \vee P}) \wedge Q\big) \Rightarrow R$$
   is true?

3. Translate the following logical statements from the formal logic language into more conventional mathematical notation. For each of the statements write its negation.
   (a) $\forall x \; ((x \in A) \Rightarrow (x \in B))$
   (b) $\forall x \; ((x \in A) \Leftrightarrow (x \in B))$
   (c) $\forall x \, \forall y \; ((x \in \mathbb{R}) \wedge (y \in \mathbb{R}) \wedge (x > y) \Rightarrow (f(x) > f(y)))$
   (d) $\forall \varepsilon > 0 \; \exists \delta > 0 \;\; \forall x \in \mathbb{R} \; (\, (0 < |x - a| < \delta) \Rightarrow |f(x) - A| < \varepsilon \,)$.

4. Translate the following statements into the formal logic notation. Using the rules of constructing negations of universal and existential statements, find their logical negations:
   (a) On some commuter train going from Montalvo to the Union Station there is a vacant seat in every car.
   (b) Every town in Sweden has a street on which at least one of the houses has all windows facing south.

5. Disprove the satements by finding a counterexample:
   (a) For every natural n the number $n^2 + n + 41$ is a prime.

(b) Every function continuous at a point has a derivative at that point.

6. Prove "De Morgan's laws": for any sets A, B and C

   (a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

   (b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

7. Prove that for any sets A, B and C

   (a) $A \cup B = A$ if and only if $B \subseteq A$

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (c) $A \setminus B = A$ if and only if $A \cap B = \emptyset$.

8. Prove that for non-empty sets A and B, $A \times B = B \times A$ if and only if $A = B$.

9. Find the subsets A and B of a set X if it is known that for every $U \subseteq X$ we have $U \cap A = U \cup B$.

10. Find the domain of $f \circ g$ and $g \circ f$ if $f(x) = \sqrt{x+1}$ and $g(x) = 1/(x-6)$.

11. Let $f(x) = x$ and $g(x) = \frac{1}{x}$. Is $f = g \circ g$?

12. Let $f : D \to \mathbb{R}$ be defined by $f(x) = (x+1)^2/x$ for $x \in D$ ($D \subseteq \mathbb{R}$ is the largest possible domain of the function). Find D, $f(D)$, $f^{-1}([1,3])$ and $f^{-1}((0,\infty))$.

13. Let $f : X \to Y$ with $A_1, A_2 \subseteq X$ and $B_1, B_2 \subseteq Y$. Prove that

   (a) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

   (b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

   (c) Give an example when $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.

   (d) Prove that if f is a one-to-one function then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

14. Suppose $f : X \to Y$ and $A \subseteq X$, $B \subseteq Y$. Prove that

   (a) $f(f^{-1}(B)) \subseteq B$. Give an example where $f(f^{-1}(B)) \neq B$

   (b) $A \subseteq f^{-1}(f(A))$. Give an example where $A \neq f^{-1}(f(A))$

   (c) f is a one-to-one function if and only if $f^{-1}(f(A)) = A$ for every $A \subseteq X$.