MATH 350 Apr. 25, 2011 (1)

# The Darboux integral

$f : [a,b] \to \mathbb{R}$ - bounded function

$P = \{ t_0, \dots t_n : a = t_0 < t_1 < \dots < t_n = b \}$

($n+1$ points, ordered by increasing.)

- subdivision of $[a,b]$ (partition)

$[a,b] = \bigcup I_i$, $I_i = [t_{i-1}, t_i]$, $i = 1 \dots n$

$M = \sup f$, $m = \inf f$ $\Delta x_i = t_i - t_{i-1}$

$M_i = \sup\limits_{I_i} f$, $m_i = \inf\limits_{I_i} f$ $i = 1 \dots n$

Def: $S^+(f,P) = \sum_{i=1}^n M_i \Delta x_i$ - upper sum on $[a,b]$

$S^-(f,P) = \sum_{i=1}^n m_i \Delta x_i$ - lower sum on $[a,b]$.

Def: $P' \subseteq P$ is a refinement of $P$.

$P_1, P_2$ - subdivisions;

Then $P_1 \cup P_2$ is a common refinement

Theorem (5.1 - Properties of upper and l. sums)

$f : [a,b] \to \mathbb{R}$ bounded,

(a) $\forall P$ - partition

$m(b-a) \leq S^-(f,P) \leq S^+(f,P) \leq M(b-a)$

(b) If $P' \subseteq P$ then

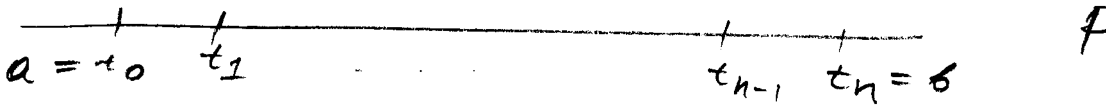
$$S^-(f,P) \le S^-(f,P') \le S^+(f,P') \le S^+(f,P)$$

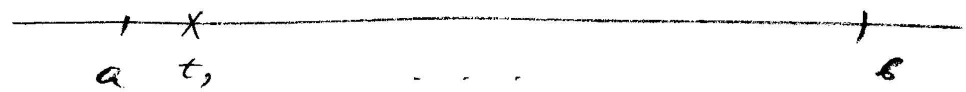
(c) If $P_1, P_2$ any two subdivisions of $[a,b]$ then

$$S^-(f,P_1) \le S^+(f,P_2)$$

<u>Proof</u>: (a) - straightforward.

(b)




$$P_i' = P' \cap I_i \quad ; \quad \text{Then } P' = \bigcup P_i'$$

$$S^-(f,P') = \sum_{j=1}^{n} S^-(f,P_j') \ge \sum_{j=1}^{n} m_i \Delta x_i = S^-(f,P)$$

$$S^+(f,P') = \sum_{j=1}^{n} S^+(f,P_j') \le \sum_{i=1}^{n} M_i \Delta x_i = S^+(f,P)$$

(c) Take $P = P_1 \cup P_2$. Then

$$S^-(f,P_1) \le S^-(f,P) \le S^+(f,P) \le S^-(f,P_2).$$

<u>Def</u>. Upper Darboux integral:

$$\overline{\int_a^b} f(x)\,dx = \inf\{S^+(f,P),\ P\text{-partition of } [a,b]\}.$$

Lower Darboux integral:

$$\underline{\int_a^b} f(x)dx = \sup\{ S^-(f,P),\ P\text{-partition of } [a,b]$$

$f$ is Darboux integrable if

$$I = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

in this case $I$ is the Darboux integral of $f$ over $[a,b]$

<u>Theorem</u> (Upper and lower bounds for the integral)

$$m(b-a) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le M(b-a)$$

<u>Proof</u>: $\forall P$-partition

$$\underbrace{m(b-a)}_{\substack{\text{lower bound}\\ \text{for}\\ \text{all lower}\\ \text{Darboux}\\ \text{sums}}} \le S^-(f,P) \le S^+(f,P) \le \underbrace{M(b-a)}_{\substack{\text{upper bound}\\ \text{for the set}\\ \text{of all upper}\\ \text{Darboux sums}}}$$

$\Rightarrow$ $m(b-a) \le \underline{\int_a^b} f(x)dx$ $\qquad \Rightarrow \overline{\int_a^b} f(x)dx \le M(b-a)$

Do it on the board carefully.

$\forall P$ $S^+(f,P)$ is an upper bound for any $S^-(f,P) \Rightarrow$

$\underline{\int_a^b} f(x)dx = \sup S^-(f,P) \le S^+(f,P)$ . . .

$\forall P$ $S^-(f,P)$ is a lower bound for the set of $S^+(f,P)$

$\Rightarrow S^-(f,P) \le \overline{\int_a^b} f(x)dx$ . . .

<u>Corollary</u>: if $f : [a,b] \to \mathbb{R}$ is integrable, $m \le f \le M$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

<u>Lemma</u>: (a) If $k > 0$ then

$$\inf kf = k \inf f, \quad \sup kf = k \sup f$$

(b) If $k < 0$ then

$$\inf kf = k \sup f, \quad \sup kf = k \inf f.$$

<u>Pf</u>: (a) let $m = \inf f$; then $\forall x\ f(x) \ge m$

$$\Rightarrow k f(x) \ge km;$$

if $km + \varepsilon$ is a lower bd then $m + \frac{\varepsilon}{k}$ is a lower bd for $f$. $\therefore$ contr.

(b) If $k < 0$, $\forall x\ f(x) \ge m$

$$\Rightarrow \forall x\ k f(x) \le km$$

$$\Rightarrow \sup k f(x) \le km$$

(Proof is an Exercise.)

<u>Theorem</u> (5.3 - Properties of Darboux integrals)

(a) If $k > 0$, $f : [a,b] \to \mathbb{R}$ bounded

$$\underline{\int_a^b} kf = k \underline{\int_a^b} f; \quad \overline{\int_a^b} kf = k \overline{\int_a^b} f$$

If $k < 0$,

$$\underline{\int_a^b} kf = k \overline{\int_a^b} f, \quad \overline{\int_a^b} kf = k \underline{\int_a^b} f$$

(b) If $h(x) = f_1(x) + f_2(x)$,

$$\underline{\int_a^b} h(x)\,dx \geq \underline{\int_a^b} f_1(x)dx + \underline{\int_a^b} f_2(x)dx$$

$$\overline{\int_a^b} h(x) \leq \overline{\int_a^b} f_1 + \overline{\int_a^b} f_2$$

(c) If $f_1 \leq f_2$ then

$$\underline{\int_a^b} f_1 \leq \underline{\int_a^b} f_2, \quad \overline{\int_a^b} f_1 \leq \overline{\int_a^b} f_2$$

(d) If $a < c < b$, $f : [a,b] \to \mathbb{R}$ bounded, then

(i) $\underline{\int_a^b} f(x) = \underline{\int_a^c} f(x) + \underline{\int_c^b} f(x)$

(ii) $\overline{\int_a^b} f(x) = \overline{\int_a^c} f(x) + \overline{\int_c^b} f(x)$

<u>Proofs</u> (a)

$$\underline{\int_a^b} kf = \sup\{\sum m_i(kf)\Delta x_i;\ x_0 \ldots x_n \in P\}$$

$$= k \sup\{\sum m_i(f)\Delta x_i,\ x_0 \ldots x_n \in P\}$$

$$= k \underline{\int_a^b} f$$

(b) $m_i(f_1 + f_2) \geq m_i(f_1) + m_i(f_2)$

$M_i(f_1 + f_2) \leq M_i(f_1) + M_i(f_2)$

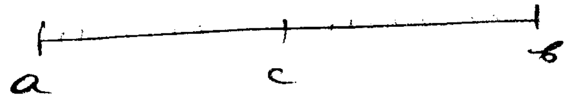
(c) $m_i(f_1) \leq m_i(f_2)$; $M_i(f_1) \leq M_i(f_2)$.

(d) $\underline{\int_a^b} f(x)dx = \sup\{S^-(f,P),\ P\text{- part of } [a,b]\}$

If $P$ includes $c$,

$S^-(f,P) = S^-(f,P_1) + S^-(f,P_2)$

$$\underline{\int_a^b} f(x)\,dx \geq \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx$$



Next, prove that

$$\underline{\int_a^c} f + \underline{\int_c^b} f = \sup\left\{ S^-(f,P);\ P\text{-part. of } [a,b] \right\}.$$

Let $P$ be such that

$$\underline{\int_a^b} f - S^-(f,P) < \varepsilon.$$

and define $P' = P \cup \{c\}$. Then

$$S^-(f,P') > S^-(f,P)$$

$$\Rightarrow \underline{\int_a^b} f - S^-(f,P') = \underline{\int_a^b} f - S^-(f,P_1) - S^-(f,P_2) < \varepsilon.$$

$\forall \varepsilon > 0$

$$\Rightarrow \underline{\int_a^b} f - \underline{\int_a^c} f - \underline{\int_c^b} f < \varepsilon$$

$$\Rightarrow \underline{\int_a^b} f \leq \underline{\int_a^c} f + \underline{\int_c^b} f.$$

$\square$

Theorem 5.4 If $f$ is integrable on $I$ then $f$ is integrable on any sub-interval $I' \subseteq I$.

Proof: If $I = \underline{\int_a^b} f = \overline{\int_a^b} f$ take $P_1, P_2$

$$I - \varepsilon \leq S^-(f,P_1) \leq I \leq S^+(f,P_2) \leq I + \varepsilon$$

...

If $f$ is bounded on $[a, b]$, then

$$\underline{\int_a^c} f + \underline{\int_c^d} f + \underline{\int_d^b} f \le \overline{\int_a^c} f + \overline{\int_c^d} f + \overline{\int_d^b} f$$

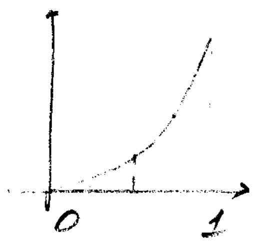
If $f$ is integrable, the above inequality becomes an equality =>

$$\underbrace{\left(\underline{\int_a^c} f - \overline{\int_a^c} f\right)}_{\ge 0} + \underbrace{\left(\underline{\int_c^d} f - \int_c^d f\right)}_{\ge 0} + \underbrace{\left(\underline{\int_d^b} f - \int_d^b f\right)}_{\ge 0} = 0$$

=> all three terms are zero

=> $f$ is integrable on each sub-interval.

Examples: 1) $f(x) = x^2$; $n = 2$, $n = 4$; $[a, b] = [0, 1]$.




$$S^-(f, P_2) = \frac{1}{8}$$

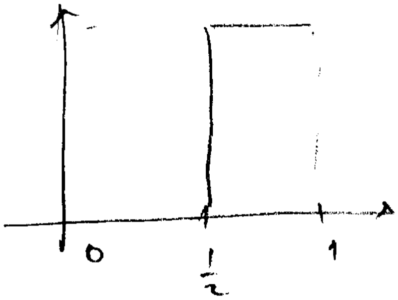
$$S^+(f, P_2) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

$$S^-(f, P_4) = 0 \cdot \frac{1}{4} + \frac{1}{16} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{9}{16} \cdot \frac{1}{4} = \frac{14}{16} \cdot \frac{1}{4} = \frac{7}{32}$$

$$S^+(f, P_4) = \frac{1}{16} \cdot \frac{1}{4} + \frac{4}{16} \cdot \frac{1}{4} + \frac{9}{16} \cdot \frac{1}{4} + \frac{16}{16} \cdot \frac{1}{4} = \frac{30}{16} \cdot \frac{1}{4} = \frac{15}{32}$$

2) $f(x) = c$ is integrable, and all upper and lower sums are the same.

3) $f(x) = 1\!\!1_{\{\frac{1}{2} \le x \le 1\}}$. integrable and has integral $\frac{1}{2}$.




$$P_\alpha = \{0, \tfrac{1}{2}+\alpha, 1\}$$

$$S^-(f, P_\alpha) = 0\cdot\left(\tfrac{1}{2}+\alpha\right) + 1\left(\tfrac{1}{2}-\alpha\right)$$

$$Q_\alpha = \{0, \tfrac{1}{2}-\alpha, 1\}$$

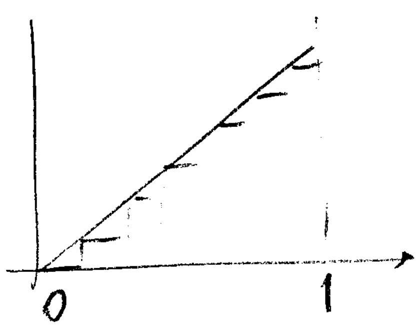
$$S^+(f, Q_\alpha) = 0\cdot\left(\tfrac{1}{2}-\alpha\right) + 1\left(\tfrac{1}{2}+\alpha\right)$$

$$\frac{1}{2} = \sup_\alpha S^-(f, P_\alpha) \le \underline{\int_a^b} f$$

$$\frac{1}{2} = \inf_\alpha S^+(f, Q_\alpha) \ge \overline{\int_a^b} f$$

$$\Rightarrow \quad \underline{\int_a^b} f = \overline{\int_a^b} f = \frac{1}{2} \quad \text{and } f \text{ is integrable}$$

4) $f(x) = x$ , $[a, b] = [0, 1]$.




$$P_n = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \dots \tfrac{n-1}{n}, 1\}$$

$$S^-(f, P_n) = \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n}\right)$$

$$= \frac{1}{n^2}\sum_{j=1}^{n-1} j$$

$$S^+(f, P_n) = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n} + \frac{n}{n}\right)$$

$$= \frac{1}{n^2}\sum_{j=1}^{n} j$$

$$S^+(f, P_n) - S^-(f, P_n) = \frac{1}{n}$$

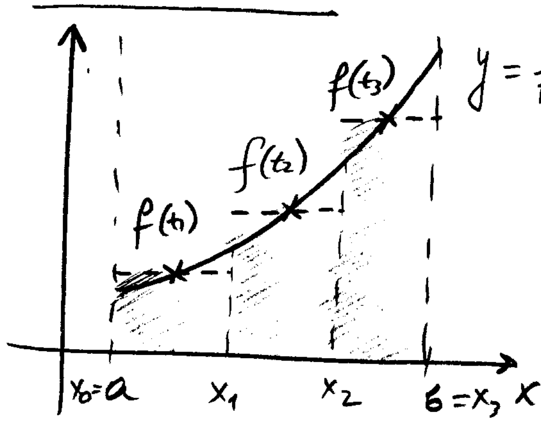
$$S^-(f, P_n) = \frac{1}{n^2}\,\frac{(n-1)n}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right) \longrightarrow \frac{1}{2}, \quad n \to \infty$$

$$S^+(f, P_n) = \frac{1}{n^2}\,\frac{n(n+1)}{2} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \longrightarrow \frac{1}{2}, \quad n \to \infty$$

5) The Dirichlet function $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ is not int on $[0, 1]$.

# The Riemann integral

**Basic idea:** to solve the area problem $I = [a,b]$.




- Subdivide the interval into $n$ sub-intervals $I_1 \ldots I_n$
- Compute values at chosen "sample points" $t_1 \ldots t_n$
- Form the "Riemann sum"

$$S(f, P, A) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

$P = \{x_0 = a, x_1 \ldots x_n = b\}$ $\quad x_0 < x_1 < \ldots < x_n$

— partition

$I_i = [x_{i-1}, x_i]$, $\Delta x_i = x_i - x_{i-1}$ - length of $I_i$

$A = \{t_1, \ldots t_n\}$ where $t_i \in I_i$

**Def:** $\|P\| = \max\limits_{i=1..n} \Delta x_i$ is the mesh size of the partition $P$.

**Def:** $f$ is Riemann integrable on $I$

$\iff \exists L \in \mathbb{R}: \ \forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall P: \|P\| < \delta$ $\forall A$

$$|S(f, P, A) - L| < \varepsilon.$$

Symbolically, $L = \lim\limits_{\|P\| \to 0} \sum_{i=1}^{n} f(t_i)\Delta x_i$

The value $L$ is called the Riemann integral of $f$ over $I$, denoted

(2)

$$L = \int_a^b f(x)dx .$$

Lemma 1 (Thm 5.9)

The Riemann integral is unique.

Pf: If $L_1 \neq L_2$ take $\varepsilon = |L_1 - L_2|/2$

Then $S(f,P,A)$ is within $\varepsilon$ from both $L_1$ and $L_2$ whenever

$$\|P\| < \delta,$$

in contradiction to triangle inequality:

$$|L_1 - L_2| \le |L_1 - S(f,P,A)| + |L_2 - S(f,P,A)|$$
$$< \frac{|L_1 - L_2|}{2} + \frac{|L_1 - L_2|}{2} = |L_1 - L_2| .$$

Lemma 2 (Thm 5.10)

If $f$ is Riemann integrable, it is bounded.

Let $\varepsilon = 1$.

Pf: Take $P, A$ such that

$$\left| \sum_{i=1}^{n} f(t_i)\Delta x_i - L \right| < 1$$

then for any $y_i \in I_i$

$$\left| \sum_{i=1}^{n} f(y_i)\Delta x_i - L \right| < 1$$

by triangle inequality. ③

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\Delta x_i - \sum_{i=1}^{n} f(y_i)\Delta x_i \right| < 2.$$

Select $y_2 = t_2,\ y_3 = t_3 \ldots\ y_n = t_n$

$$\Rightarrow |(f(t_1) - f(y_1))\Delta x_1| < 2$$

$$\Rightarrow |f(y_1)| < |f(t_1)| + \frac{2}{\Delta x_1}$$

Similarly $|f(y_i)| < |f(t_i)| + \frac{2}{\Delta x_i}$

$$\Rightarrow \forall y \in I \quad |f(y)| < \max_{i=1..n} \left\{ |f(t_i)| + \frac{2}{\Delta x_i} \right\}$$

<u>Theorem</u> (5.12 : Riemann $\Leftrightarrow$ Darboux)

$f : [a,b] \to \mathbb{R}$ bounded.

Then $f$ is Riemann int. $\Leftrightarrow$ $f$ is Darboux int.

and the two values of integral are the same:

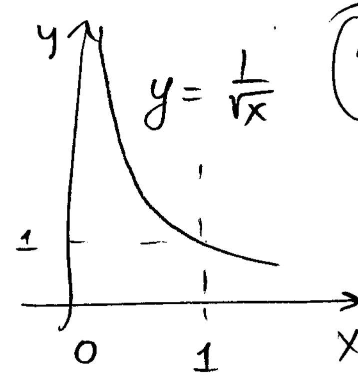
$$L = \int_a^b f(x)\,dx.$$

<u>Recall</u>: $f$ is Darboux integrable $\Leftrightarrow$ upper Darboux integral = lower Darboux integral.

Upper Darboux int is the infimum of upper sums.

Lower Darboux int is the supremum of lower sums.

<u>Example</u>: $f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x>0 \\ 0, & x=0 \end{cases}$




is unbounded on $[0, 1]$

$\Rightarrow$ is <u>not</u> Riemann integrable.

Note that $f$ is integrable in a "generalized" sense (through improper integrals)

Indeed, $$\int_0^1 f(x)dx = \int_0^1 \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]_0^1 = 2$$

and this can be made rigorous by treating

$$\int_0^1 f(x)dx = \lim_{a\to 0} \int_a^1 \frac{1}{\sqrt{x}} dx = \lim_{a\to 0} 2(1-\sqrt{a}).$$

Now, what's the problem in defining the integral through Darboux / Riemann sums?

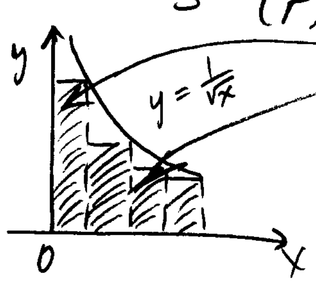
Darboux has obvious problem that for any partition $P$ of $[0,1]$

$$M_1(f) = \sup_{x\in I_1} f(x) = +\infty, \quad \text{so} \quad S^+(f,P) = +\infty$$

for any partition

Lower sums seem to be OK: Take $P = P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots 1\}$

$$S^-(f, P_n) = \frac{1}{n}\left(\sqrt{n} + \sqrt{\tfrac{n}{2}} + \sqrt{\tfrac{n}{3}} + \dots + \sqrt{\tfrac{n}{n-1}} + 1\right)$$




$$\le \underbrace{\frac{1}{\sqrt{n}}}_{\text{(first cell)}} + \underbrace{\int_{\frac{1}{n}}^1 \frac{1}{\sqrt{x}} dx}_{\text{(remaining cells)}} = \frac{1}{\sqrt{n}} + 2\left(1 - \frac{1}{\sqrt{n}}\right) = 2 - \frac{1}{\sqrt{n}}$$

$\to 2, \ n\to\infty.$

However, a careless use of Riemann sums could produce a wrong answer. 5

Take $P = P_n = \{0, \frac{1}{n}, \frac{2}{n} \dots 1\}$ and let

$t_1 = \frac{1}{n^3} \in I_1$, and $t_i \in I_i$, $i = 2 \dots n$ are arbitrary.

Then

$$S(f, P, A) = \frac{1}{n} \cdot n^{3/2} + \frac{1}{n} f(t_2) + \dots + \frac{1}{n} f(t_n)$$

$$\geq \sqrt{n} \longrightarrow \infty, \quad n \to \infty$$

Now the contribution from the first cell is too large

Clearly, the Darboux / Riemann approaches are not satisfactory when it comes to treatment of unbounded functions. This served as one of the motivations (perhaps not the most important one) for the development of a more general concept of integral by Lebesgue.

<u>Theorem</u> (5.5 - Criterion of Darboux integrability)

Let $f : [a,b] \to \mathbb{R}$ be bounded.

Then $f$ is integrable $\iff$

$\forall \varepsilon > 0 \ \exists P$ :

$$S^+(f,P) - S^-(f,P) < \varepsilon.$$

<u>Proof</u>: "$\Rightarrow$" Let $f$ be integrable. Then

$$I = \sup S^-(f,P) = \inf S^+(f,P)$$

$\Rightarrow \exists P_1 : \quad I - S^-(f,P_1) < \frac{\varepsilon}{2}$

$\exists P_2 : \quad S^+(f,P_2) - I < \frac{\varepsilon}{2}$

$$\Rightarrow \quad S^+(f,P_1) - S^-(f,P_2) < \varepsilon.$$

Let $P = P_1 \cup P_2 \Rightarrow$

$$S^+(f,P) - S^-(f,P) < \varepsilon.$$

"$\Leftarrow$"

$$\underline{\int_a^b} f(x)\,dx = \sup S^-(f,P)$$

$$\overline{\int_a^b} f(x)\,dx = \inf S^+(f,P)$$

Take $P$ which satisfies the $\varepsilon$-condition.

Then

$$S^-(f,P) \le \underline{\int_a^b} f(x)\,dx$$

$$S^+(f,P) \ge \overline{\int_a^b} f(x)\,dx$$

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le S^+(f,P) - S^-(f,P) < \varepsilon$$

Since $\varepsilon$ is arbitrary the upper and lower integrals are equal, and $f$ is integrable

<u>Theorem</u> If $f : [a, b] \to \mathbb{R}$ is continuous then $f$ is integrable.

(Cor. 1 to Thm 5.5)

<u>Proof</u>: Use in a crucial way that $f$ is uniformly continuous.

$\forall \varepsilon > 0 \quad \exists \delta > 0$:

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \text{if} \quad |x-y| < \delta.$$

Take $P$ such that $\max \Delta x_i < \delta$.

$$S^+(f, P) = \sum_{i=1}^n M_i(f) \Delta x_i = \sum_{i=1}^n f(y_i) \Delta x_i$$

(by extreme value theorem, sup is achieved at $y_i \in I_i$)

$$S^-(f, P) = \sum_{i=1}^n m_i(f) \Delta x_i = \sum_{i=1}^n f(z_i) \Delta x_i$$

$$S^+(f,P) - S^-(f,P) = \sum_{i=1}^n (f(y_i) - f(z_i)) \Delta x_i$$

$$\leq \sum_{i=1}^n |f(y_i) - f(z_i)| \Delta x_i$$

$$\leq \sum_{i=1}^n \frac{\varepsilon}{b-a} \Delta x_i = \frac{\varepsilon}{b-a} \cdot b-a = \varepsilon. \quad \square$$

<u>Theorem</u> (5.6 – Mean-value theorem for integrals.)
$f : [a, b] \to \mathbb{R}$ continuous.

Then $\exists c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

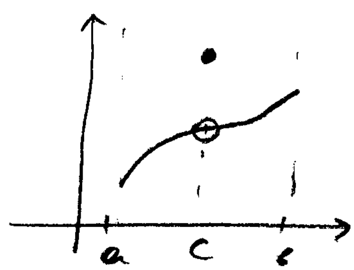
Proof: We have

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$\Rightarrow \quad m \le \underbrace{\frac{1}{b-a}\int_a^b f(x)dx}_{=A} \le M$$

By IVT and/or EVT $\exists c \in [a,b] \; : \; f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

$$\Rightarrow \int_a^b f(x)dx = f(c)(b-a). \qquad \boxtimes$$

#17: 

Use criterion of integrability.

#18: Use localization lemma for continuous functions.

#19 Choose $g$ localized.

# Integration, further developments

## Fundamental theorem of calculus.

Thm (5.7 FTC-I)

$f : [a,b] \to \mathbb{R}$ continuous,

$$F(x) = \int_a^x f(t)\,dt$$

Then $F$ is continuous on $[a,b]$ and $F'(x) = f(x)$, $x \in (a,b)$.

Proof: let $x \in (a,b)$. If $h > 0$,

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt = f(\xi)h,$$
$$\xi \in (x, x+h)$$

if $h < 0$,

$$F(x+h) - F(x) = -\int_{x+h}^{x} f(t)\,dt =: \int_x^{x+h} f(t)\,dt$$
$$= f(\xi)h,$$
$$\xi \in (x+h, x)$$

$\Rightarrow F$ is differentiable at $x$ and $F'(x) = f(x)$

Similarly can check continuity and left / right derivatives at $x = a, b$.

Rmk: Can obtain the same result using

$$\left(\min_{[x,x+h]} f\right)|h| \le \int_x^{x+h} f(t)\,dt \le \left(\max_{[x,x+h]} f\right)|h|$$

Thm (5.8 FTC-II).

$f, F : [a,b] \to \mathbb{R}$ continuous,

$F'(x) = f(x)$ on $(a,b)$.

Then $\int_a^b f(x)\,dx = F(b) - F(a)$

Proof: Let $G(x) = \int_a^x f(t)\,dt$; $G(a) = 0$.

Then $\int_a^b f(x)\,dx = G(b) - G(a)$.

WTS $F(x) = G(x) + C$, then

$$F(b) - F(a) = G(b) + C - G(a) + C = G(b) - G(a)$$
$$= \int_a^b f(x)\,dx.$$

Let $u(x) = F(x) - G(x)$; prove that $u = C$.

$u$ is differentiable on $(a,b)$ and

$u'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$

If $\exists\, x_1 \neq x_2$ such that $u(x_1) \neq u(x_2)$

then by Lagrange's mean value theorem

$$\exists c : \quad u'(c) = \frac{u(x_2) - u(x_1)}{x_2 - x_1} \neq 0$$

a contradiction.

Thus, $u(x)$ has the same value throughout $(a,b) \Rightarrow$ by continuity on $[a,b]$. $\square$

Problem: If $f, g$ are integrable on $[a,b]$ show that $F = \max\{f, g\}$ is integrable.

Key estimate:

$$\sup F - \inf F \leq \max\{\sup f - \inf f, \sup g - \inf g\}$$

$$\sup F = \max\{\sup f, \sup g\}$$
$$\inf F \geq \max\{\inf f, \inf g\}.$$

$$\Rightarrow \sup F - \inf F \leq \sup f - \inf f$$

and

$$\sup F - \inf F \leq \sup g - \inf g$$

This proves the key estimate.

Now, if $f$ is integrable, $\exists P_1$ - partition of $[a,b]$ such that

$$S^+(f, P_1) - S^-(f, P_1) = \sum_{i=1}^{n} (M_i(f) - m_i(f)) \Delta x_i < \frac{\varepsilon}{2}$$

similarly, $\exists P_2$ :

$$S^+(g, P_2) - S^-(f, P_2) = \sum_{i=1}^{n} (M_i(g) - m_i(g)) \Delta x_i < \frac{\varepsilon}{2}$$

Take $P$ to be the common refinement of $P_1, P_2$. Then

$$S^+(F, P) - S^-(F, P) = \sum_{i=1}^{n} (M_i(F) - m_i(F)) \Delta x_i$$

$$\leq \sum_{i=1}^{n} (M_i(f) - m_i(f)) \Delta x_i + \sum_{i=1}^{n} (M_i(g) - m_i(g)) \Delta x_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

<u>Corollary</u>: If $f$ is integrable, then the following are integrable:

$$|f| = \max\{f(x), -f(x)\}$$
$$f_+ = \max\{f(x), 0\}$$
$$f_- = \max\{-f(x), 0\}.$$

#13 (b) Show that
$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Since $-|f(x)| \leq f(x) \leq |f(x)|$

$$\Rightarrow -\int_a^b |f|\,dx \leq \int_a^b f\,dx \leq \int_a^b |f|\,dx$$

$$\Rightarrow \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f|\,dx.$$

## Proof of Theorem 5.12

<u>Thm</u>: $f : [a, b] \to \mathbb{R}$ bounded

Then $f$ is Riemann int. $\Leftrightarrow$ $f$ is Darboux integrable (upper int = lower int.)

and if $L$ is the Riemann int. of $f$ then

$$L = \int_a^b f(x)dx.$$

Part I:

<u>Theorem</u> (5.11: Riemann $\Rightarrow$ Darboux)

$f$ - Riemann int. on $[a, b]$. Then $f$ is Darboux int and $L = \int_a^b f(x)dx$.

Proof: $f$ is Riemann integrable:

$$\exists L \in \mathbb{R} : \quad \forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall P, A, \ \|P\| < \delta$$

$$\left| \sum_{i=1}^{n} f(t_i) \Delta x_i - L \right| < \frac{\varepsilon}{4}$$

Take $t_i' \in I_i$ such that $M_i(f) - f(t_i') < \frac{\varepsilon}{4(b-a)}$

$t_i'' \in I_i$ such that $f(t_i'') - m_i(f) < \frac{\varepsilon}{4(b-a)}$

Then

$$S^+(f,P) - S^-(f,P) = \sum_{i=1}^{n} \left(M_i(f) - m_i(f)\right) \Delta x_i$$

$$= \sum_{i=1}^{n} \left(M_i(f) - f(t_i')\right) \Delta x_i + \sum_{i=1}^{n} f(t_i') \Delta x_i - L$$

$$+ L - \sum_{i=1}^{n} f(t_i'') \Delta x_i + \sum_{i=1}^{n} \left(f(t_i'') - m_i(f)\right) \Delta x_i$$

$$< \frac{\varepsilon}{4} + \left| \sum_{i=1}^{n} f(t_i') \Delta x_i - L \right| + \left| L - \sum_{i=1}^{n} f(t_i'') \Delta x_i \right| + \frac{\varepsilon}{4}$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Part II

[Good results for upper and lower sums are always obtained for partitions with small mesh.]

Lemma (5.2) $f : [a,b] \to \mathbb{R}$ is bounded.

Then $\forall \varepsilon > 0 \ \exists \delta > 0 ; \ \forall P \ \|P\| < \delta \Rightarrow$

$$S^+(f,P) < \overline{\int_a^b} f(x)\,dx + \varepsilon \quad \text{and}$$

$$S^-(f,P) > \underline{\int_a^b} f(x)\,dx - \varepsilon$$

Proof: For the upper sums:

Take $P_0 = \{x_0, x_1 \ldots x_{n_0}\}$ such that

$$S^+(f, P_0) = \sum_{i=1}^{n} M_i(f)\,\Delta x_i < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

Choose $\delta < \min \Delta x_i$; then $\forall P : \|P\| < \delta$, there are at most $n_0 - 1$ subintervals $I_i$ for which $x_i \in I_i$ ($x_i$ are from $P_0$, $I_i$ are from $P$ ...)

let these intervals be $I_{i_1} \ldots I_{i_p}$ and the remaining ones are $I_{i_{p+1}} \ldots I_{i_n}$

Then

$$S^+(f, P) = \sum_{k=1}^{p} M_{i_k}(f)\,\Delta x_{i_k} + \sum_{k=p+1}^{n} M_{i_k}(f)\,\Delta x_{i_k}$$

$$\le p M \delta + S^+(f, P_0)$$

$$< \int_a^b f(x)\,dx + \frac{\varepsilon}{2} + pM\delta \le \int_a^b f(x)\,dx + \varepsilon$$

if $\delta < \frac{\varepsilon}{pM}$.

Similarly, we can find $\delta_2$ for the lower sums, then the smaller of $\delta_2$ and the previously obtained $\delta$ would work for both.

<u>Proof of Thm 5.12</u> Assume that $f$ is Darboux integrable.

Then $\forall \varepsilon > 0$ pick $\delta > 0$ as in the prev. Lemma

$\Rightarrow \forall P : \|P\| < \delta$ we have

$$\int_a^b f(x)dx - \varepsilon < S^-(f,P) \le S(f,P,A) \le S^+(f,P) < \int_a^b f(x)dx + \varepsilon$$

$$\Rightarrow \quad \left| S(f,P,A) - \int_a^b f(x)dx \right| < \varepsilon$$

$\Rightarrow \int_a^b f(x)dx$ is the Riemann integral of $f$. $\square$