Mon, Apr 28, 2014 **MATH 340, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

This test includes 7 questions (total of 38 points + 2 points bonus), on 7 pages. The last page contains a table of the standard normal distribution. The duration of the test is 1 hr 15 min.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. If a numerical answer is required, acceptable forms are a reduced fraction or a decimal with sufficient accuracy. Show all steps.

1. (6 points) An airline estimates that 4 percent of the passengers that make reservations on a particular flight do not show up. Consequently, their policy is to sell 100 reserved seats on a plane that has only 98 seats. Using binomial distribution, calculate the probability that every person who shows up for the flight will find a seat available.

X = number of 'no-show' passengers.

$X \sim \text{Binomial}(100, 0.04)$

(assume each passenger shows up independently of the others...)

{Every passenger finds a seat} = {at least two 'no shows'}

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - 0.96^{100} 0.04^{0} - \binom{100}{1} 0.96^{99} \cdot 0.04^{1}$$

$$\approx \underline{0.9128}$$

2. (6 points) Suppose that the average number of accidents on a stretch of a highway on a given day is 2.4. Approximate the probability that there will be

   (a) at most one accident today

   (b) three or more accidents during the next two days.

   Explain your reasoning.

(a) Large number of cars (n);
each has small prob. to have accident (p)

$$np \approx 2.4$$

$$X \sim \text{Poisson}(\lambda = 2.4)$$

$$P(X \le 1) = P(X=0) + P(X=1) = e^{-2.4} + 2.4e^{-2.4}$$

$$= 3.4e^{-2.4} \approx \underline{0.3084}$$

(b) During two days, twice the number of cars;

$$\lambda \approx 2np \approx 4.8$$

$$X \sim \text{Poisson}(\lambda = 4.8)$$

$$P(X \ge 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - e^{-4.8}\left(1 + 4.8 + \frac{4.8^2}{2}\right)$$

$$\approx \underline{0.8575}$$

Continued...

3. (6 points) The probability density function of a random variable X is given by

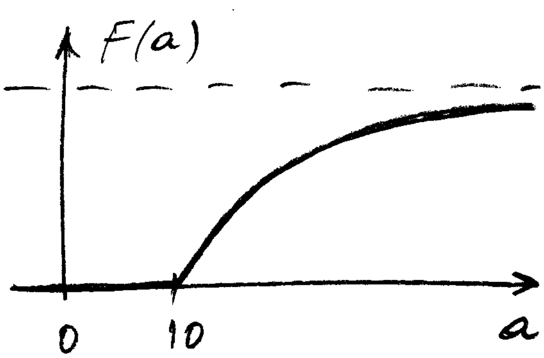
$$f(x) = \begin{cases} 10x^{-2}, & x > 10 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P(X > 20)$

$$P(X>20) = \int_{20}^{\infty} 10x^{-2}\,dx = \left[-\frac{10}{x}\right]_{20}^{\infty} = \frac{1}{2}$$

(b) Find an expression for the cumulative distribution function of X and sketch its graph.

$$F(a) = \int_{-\infty}^{a} f(x)\,dx = \begin{cases} 0, & a < 10 \\ \int_{10}^{a} 10x^{-2}\,dx, & a \geq 10 \end{cases}$$



$$= \begin{cases} 0, & a < 10 \\ 1 - \frac{10}{a}, & a \geq 10 \end{cases}$$

$$= \max\left\{0, 1 - \frac{10}{a}\right\}.$$

(c) (bonus: 2 points) Find $\mathbb{E}[X]$ and the median of X.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{10}^{\infty} 10x^{-1}\,dx$$

$$= \left[10 \ln x\right]_{x=10}^{\infty} = \infty$$

(expectation is infinite!)

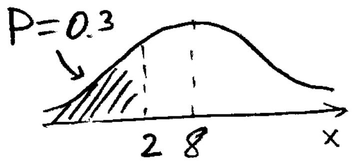
Median is the value that separates the range of X into two regions of probability $\frac{1}{2}$.

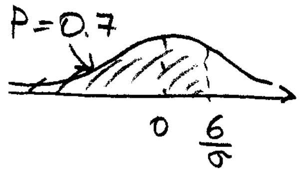
Continued...

By part (a), median $= 20$. OR: $F(a) = \frac{1}{2}$

$1 - \frac{10}{a} = \frac{1}{2} \Rightarrow a = 20$.

4. (4 points) Suppose X is a normal random variable with mean 8. If $P(X < 2) = 0.3$, approximately what is $\mathrm{Var}(X)$?





$$P(X<2) = 0.3$$

$$P(X-8<-6) = 0.3$$

$$P\left(\frac{X-8}{\sigma} < -\frac{6}{\sigma}\right) = 0.3$$

$$P\left(Z < -\frac{6}{\sigma}\right) = 0.3$$

$$P\left(Z > \frac{6}{\sigma}\right) = 0.3$$

$$P\left(Z \leq \frac{6}{\sigma}\right) = 0.7$$

$$\frac{6}{\sigma} \approx 0.525$$

$$\sigma \approx \frac{6}{0.525} \approx 11.4 \,;\ \sigma^2 \approx 130.6$$

5. (4 points) Twelve percent of the population is left-handed. Approximate the probability that in a school of 200 students there are at least 20 left-handers. State your assumptions.

Assume: Each student is left-handed with probability $p = 0.12$, independently of the others.

X - number of left-handed students.

$X \sim \text{Binomial}(200, 0.12)$ $\quad np = 24$, $npq = 21.12$

$$P(X \geq 20) = P(X \geq 19.5) = P\left(\frac{X-24}{\sqrt{21.12}} \geq \frac{19.5-24}{\sqrt{21.12}}\right)$$

$$\approx P(Z \geq -0.98) = P(Z \leq 0.98)$$

$$= \Phi(0.98) \approx 0.8365.$$

Continued...

6. (6 points)

(a) Let X be an exponential random variable. Show that $P(X > s+t \,|\, X > t) = P(X > s)$ for $s > 0$, $t > 0$ (the "memoryless property").

$X \sim \text{Exponential}(\lambda) \Rightarrow P(X > a) = e^{-\lambda a}$

$$P(X > s+t \mid X > t) = \frac{P(X > s+t)}{P(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = P(X > s).$$

(b) Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. If a car has been in the shop for 2 hours already, what is the probability that the total repair time exceeds 4 hours?

$X \sim \text{Exponential}\left(\lambda = \frac{1}{2}\right)$

$$P(X > 4 \mid X > 2) = P(X > 2) = e^{-\frac{1}{2}\cdot 2} = e^{-1}$$

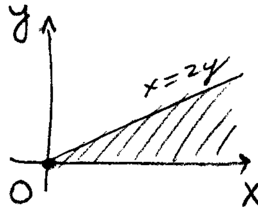
$$\approx 0.3679$$

7. (6 points) The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x > 0,\ y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P(X > 2Y)$

$$P(X>2Y) = \iint_{\left\{\substack{x>0,\,y>0 \\ x>2y}\right\}} e^{-(x+y)}\,dx\,dy = \int_0^\infty \int_{2y}^\infty e^{-x} e^{-y}\,dx\,dy$$



$$= \int_0^\infty e^{-y} e^{-2y}\,dy = \int_0^\infty e^{-3y}\,dy$$

$$= \left[-\frac{e^{-3y}}{3}\right]_0^\infty = \frac{1}{3}.$$

(b) Find $\mathbb{E}[XY]$

$$\mathbb{E}[XY] = \int_0^\infty \int_0^\infty xy e^{-(x+y)}\,dx\,dy = \left(\int_0^\infty x e^{-x}\,dx\right)^2 = 1^2 = 1.$$

(c) Find the marginal densities $f_X(x)$ and $f_Y(y)$. Are X and Y independent? Justify your answer.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_0^\infty e^{-(x+y)}\,dy, & x>0 \\ 0, & x<0 \end{cases}$$

$$= \begin{cases} e^{-x}, & x>0 \\ 0, & x<0 \end{cases}$$

By symmetry, $f_Y(y) = \begin{cases} e^{-y}, & y>0 \\ 0, & y<0. \end{cases}$

$f(x,y) = f_X(x) f_Y(y) \Rightarrow$ X and Y are independent.

The end.

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998