Wed, Mar 5, 2014 **MATH 340, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

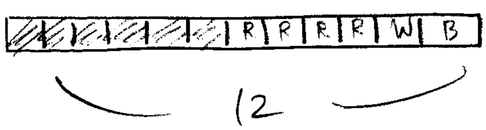
This test includes 8 questions (total of 48 points), on 8 pages. The perfect score is 42; one question counts as a bonus. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) A child has 12 blocks, of which 6 are black, 4 are red, 1 is white and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?



$12!$ arrangements if every block is given a unique label.

$6!\,4!\,1!\,1!$ ways to arrange blocks of same color once an arrangement of colors is chosen

$$\Rightarrow \quad N \cdot 6!\,4! = 12!$$

$$\Rightarrow \quad N = \frac{12!}{6!\,4!} = 27{,}720$$

2. (6 points) (a) List the three axioms of probability, in the case of a sample space with finitely many outcomes.

1. $0 \leq P(E) \leq 1$ if E is any event in S. (sample space)

2. $P(S) = 1$

3. If $E_1 \ldots E_n$ are mutually exclusive,

$$P(E_1 \cup \ldots \cup E_n) = P(E_1) + \ldots + P(E_n).$$

(in fact, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ is sufficient if S is finite.)

(b) Prove, based on these axioms, that for any events A, B

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

$$B = \underbrace{BA \cup BA^c}_{\text{mutually exclusive}}$$

$$A \cup B = \underbrace{A \cup BA^c}_{\text{mutually exclusive}}$$

Axiom 3 $\Rightarrow$ $\begin{cases} P(B) = P(BA) + P(BA^c) \\ P(A \cup B) = P(A) + P(BA^c) \end{cases}$ subtract $\Rightarrow$

$$P(A \cup B) - P(B) = P(A) - P(BA)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(AB).$$

3. (6 points) Assume that A and B are two events with positive probabilities.

(a) Prove that if $P(A|B) > P(A)$ then $P(B|A) > P(B)$.

$$P(A|B) = \frac{P(AB)}{P(B)} > P(A)$$

(Since $P(A), P(B) > 0$)

$$\Leftrightarrow \quad P(AB) > P(B)\,P(A)$$

$$\Leftrightarrow \quad \frac{P(AB)}{P(A)} > P(B)$$

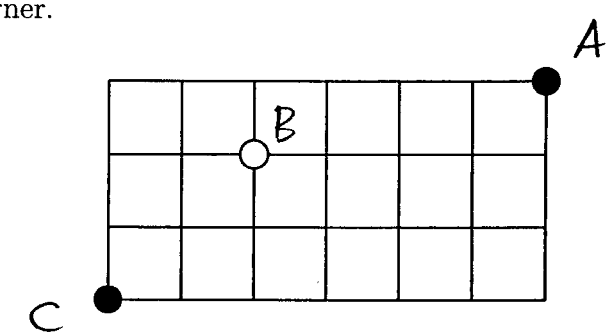
$$\Leftrightarrow \quad P(B|A) > P(B)$$

(b) If $P(A^c) \neq 0$ and $P(B|A) = P(B|A^c)$ prove that the events A and B are independent.

$$\begin{aligned} P(B) &= P(BA) + P(BA^c) \\ &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= P(B|A)P(A) + P(B|A)(1-P(A)) \\ &= P(B|A) \end{aligned}$$

$\Rightarrow$ B, A are independent.

4. (6 points) Consider the grid of points shown in the figure. Suppose that, starting at the top right corner, you can go one step down or one step left at each move, until you reach the bottom left corner.



(a) How many different paths are there that go through the point marked by the white circle?

$A \to B$: 5 steps in total, 1 step down

$\binom{5}{1}$ paths

$B \to C$: 4 steps in total, 2 steps down

$\binom{4}{2}$ paths

$A \to B \to C$: $\binom{5}{1} \cdot \binom{4}{2} = 30$ paths (basic principle of counting)

(b) If a path through the grid is chosen at random, what is the probability that it will pass through the marked point?

$A \to C$: 9 steps in total, 3 steps down.

$$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 3 \cdot 28 = 84 \text{ paths}$$

$$P = \frac{30}{84} = \frac{5}{14} \approx 0.3571$$

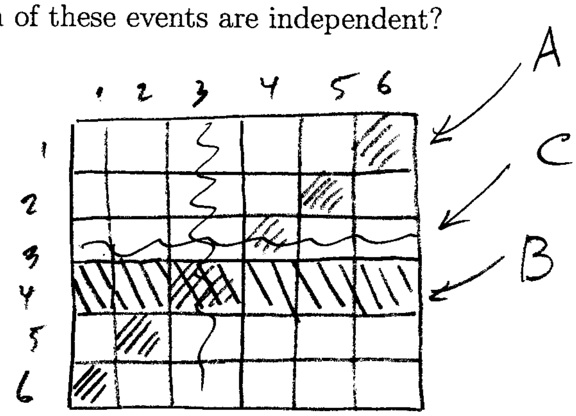
if a path is chosen "at random".

Continued...

5. (6 points) A fair die is rolled twice. Consider the events

$$A = \{\text{"sum of the dice is 7"}\}$$
$$B = \{\text{"first die lands on 4"}\}$$
$$C = \{\text{"at least one die lands on 3"}\}.$$

Which of these events are independent?



$P(A) = \frac{1}{6}$

$P(B) = \frac{1}{6}$

$P(C) = \frac{11}{36}$

$P(AB) = \frac{1}{36} = P(A)P(B)$

$P(BC) = \frac{1}{36} \neq P(B)P(C)$

$P(AC) = \frac{1}{18} \neq P(A)P(C)$

$\Rightarrow$ only the pair A, B is independent.

(B,C; A,C; A,B,C – not independent)

6. (6 points) One urn contains 2 white and 4 black balls, while another urn contains 3 white balls and 3 black balls. A ball is chosen at random from the first urn and put into the second one. A ball is then randomly selected from the second urn.

(a) What is the probability that the ball selected from the second urn is white?

$$P(W_2) = P(W_2 | W_1) P(W_1) + P(W_2 | W_1^c) P(W_1^c)$$

$$= \frac{4}{7} \cdot \frac{2}{6} + \frac{3}{7} \cdot \frac{4}{6} = \frac{20}{42} = \frac{10}{21}$$

$$\simeq 0.4762$$

(b) What is the conditional probability that the transferred ball was white given that a white ball was selected from the second urn?

$$P(W_1 | W_2) = \frac{P(W_2 | W_1) P(W_1)}{P(W_2)} = \frac{\frac{4 \cdot 2}{7 \cdot 6}}{\frac{20}{7 \cdot 6}}$$

$$= \frac{8}{20} = \frac{2}{5} = 0.4$$

7. (6 points) If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt *a three of a kind*? (This occurs when the cards have denominations a, a, a, b, c where a, b and c are distinct.)

$$P(\text{"three of a kind"}) = \frac{\underbrace{\binom{13}{3}}_{\text{choose denom}} \underbrace{\binom{3}{1}}_{\text{choose "a"}} \underbrace{\binom{4}{3}}_{\text{choose suit for "a"}} \underbrace{\binom{4}{1}\binom{4}{1}}_{\text{choose suits for "b" and "c"}}}{\binom{52}{5}}$$

$$= \frac{\frac{13\cdot 12\cdot 11}{\cancel{1\cdot 2\cdot 3}} \cdot 3\cdot 4\cdot 4\cdot 4}{\frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{\cancel{1\cdot 2\cdot 3}\cdot 4\cdot 5}}$$

$$= \frac{\cancel{13}\cdot\cancel{12}\cdot 11\cdot 3\cdot \cancel{4}\cdot\cancel{4}\cdot 4\cdot 4\cdot 5}{\cancel{52}\cdot 51\cdot 50\cdot 49\cdot \cancel{48}}$$

$$= \frac{11\cdot 8}{17\cdot 49\cdot 5} \approx 0.02113$$

8. (6 points) An urn contains w white balls, b black balls, and r red balls. Find the probability of a white ball being drawn before a black ball, if

   (a) Each ball is replaced after being drawn

   (b) No balls are replaced.

The key to the problem is to realize that the probability in this case does not depend on the number of red balls. If $r=0$, then clearly $P = \frac{w}{w+b}$ in both cases. However, statement of intuition does not present a valid mathematical argument, and this fact has to be justified in each case.

(a) Let $E = \{\text{"white before black"}\}$

Then $P(E) = 1 \cdot \frac{w}{w+b+r} + 0 \cdot \frac{b}{w+b+r} + P(E)\frac{r}{w+b+r}$

(condition on the outcome of the first ball drawn.)

$$\Rightarrow \quad \frac{w+b}{w+b+r} P(E) = \frac{w}{w+b+r} \Rightarrow P(E) = \frac{w}{w+b}$$

(same answer can be obtained by counting the number of r's before "w" or "b" and summing the geometric series, as in Example 3.4h)

(b) Let P_r be the probability of "white before black" with r red balls in the urn.

Then

$$P_r = 1 \cdot \frac{w}{w+b+r} + 0 \, \frac{b}{w+b+r} + P_{r-1} \frac{r}{w+b+r}$$

Since $P_0 = \frac{w}{w+b}$, we find recursively

$$P_1 = \frac{w}{w+b+r} + \frac{w}{w+b} \, \frac{r}{w+b+r} = \frac{w(w+b+r)}{(w+b)(w+b+r)}$$

$$= \frac{w}{w+b}$$

$$P_2 = \frac{w}{w+b}, \text{ etc.}$$

(induction can be used to show it for every r.)

The case without replacement also has a combinatorial solution:

choose positions for red balls

choose position for first white ball

Choose positions for remaining $w-1$ balls.

$$P_r = \frac{\binom{w+r+b}{r} \cdot 1 \cdot \binom{w+b-1}{w-1}}{\binom{w+r+b}{w,\, r,\, b}} = \frac{w}{w+b}.$$

ways to distribute colors on a sequence of $w+r+b$ balls.