May 13, 2013

MATH 340: Final exam review questions

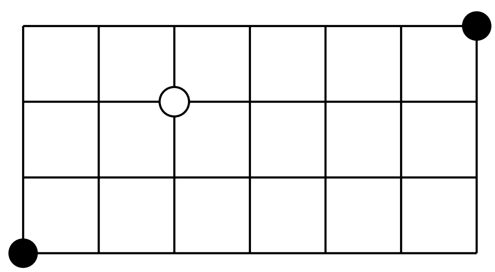
1. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many different bridge deals are possible?

2. A poker hand is said to be a *flush* if all five cards are of the same suit.

   (a) If five cards are selected at random what is the probability of being dealt a flush?

   (b) What is the conditional probability of being dealt a flush if the first two cards dealt are spades?

3. Consider the grid of points shown in the figure. Suppose that, starting at the top right corner, you can go one step down or one step left at each move, until you reach the bottom left corner.



   (a) How many different paths from one corner to another are possible?

   (b) How many different paths are there that go through the point marked by the white circle?

4. List the three axioms of probability. Show, using these axioms, that for any events A, B and C

   (a) $P(A) = P(AB) + P(AB^c)$

   (b) $P(A \cup B) = P(A) + P(B) - P(AB)$

   (c) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$.

5. **(2.33)** A forest contains 20 elk, of which 5 are captured, tagged and then released. A certain time later, 4 of the 20 elk are captured. What is the probability that 2 of these 4 have been tagged?

6. Assume that E and F are two events with positive probabilities. Show that if $P(E|F) = P(E)$ then $P(F|E) = P(F)$.

7. One coin in a collection of 65 has two heads. The rest are fair. If a coin, chosen at random from a lot and then tossed, turns up heads 6 times in a row, what is the probability that it is a two-headed coin?

8. **(4.42)** Suppose that when in flight, airplane engines will fail with probability $1-p$ independently from engine to engine. If an airplane needs a majority of its engines operative to make a successful flight, for what values of p is a 5-engine plane preferable to a 3-engine plane?

9. A point X is picked at random (with uniform density) in the interval $(0,1)$. Find the probability that $X > 1/2$, given that

   (a) $X > 1/4$

   (b) $|X - 1/2| < 1/4$.

10. Suppose you toss a dart at a circular target of radius 10 inches. Given that the dart lands in the upper half of the target, find the probability that

    (a) it lands in the right half of the target

    (b) its distance from the center is less than 5 inches

    (c) its distance from the center is more than 5 inches

    (d) it lands within 5 inches of the point $(0,5)$ (the origin is the center of the circle).

11. Let X be a random variable with density function

$$f_X(x) = \begin{cases} cx(1-x), & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

    (a) What is the value of c?

    (b) What is the cumulative distribution function of X?

    (c) What is the probability that $X < 1/4$?

12. Suppose that the cumulative distribution function of X is given by

$$F(a) = \begin{cases} 0, & a < 0 \\ 1/8, & 0 \leq a < 1 \\ 1/4, & 1 \leq a < 2 \\ 7/8, & 2 \leq a < 4 \\ 1, & a \geq 4. \end{cases}$$

    (a) Find $P(X = i)$ for $i = 0, 1, 2, 3, 4$.

    (b) Find $\mathbb{E}[X]$.

13. The cumulative distribution function of Z is given by

$$F_Z(z) = \begin{cases} 0, & z < a \\ (z-a)/(b-a), & a \leq z \leq b \\ 1, & z > b, \end{cases}$$

    where a and b are given numbers.

    (a) Find the probability density of X.

(b) Find $\mathbb{E}[X]$.

(c) Find $\text{Var}(X)$.

14. Let U be uniformly distributed on $(0, 1)$. What is the probability that the equation

$$x^2 + 4Ux + 1 = 0$$

has two real roots, x_1 and x_2?

15. Suppose that the number of years a car will run is exponentially distributed with parameter $\lambda = 1/4$. If a buyer purchases a used car which is one year old today, what is the probability that it will still run after 4 years?

16. Let X be random variable normally distributed with parameters $\mu = 70$, $\sigma = 10$. Estimate

   (a) $P(X > 50)$

   (b) $P(X > 60)$

   (c) $P(|X - 60| > 20)$.

17. A card is drawn at random from a deck of playing cards. If a card is a spade or a club, the player wins 1 dollar; if it is a heart or a diamond, the player loses 2 dollars. Find the expected value of the game.

18. An electronic component has lifetime X which is distributed according to the probability density

$$f(x) = cx^2e^{-x}, \quad x > 0,$$

where c is a constant.

   (a) Determine the value of the constant c.

   (b) Compute the expected lifetime of the device.

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19. **(5.18)** Suppose that X is a normal random variable with mean 5. If $P(X > 9) = 0.2$, approximately what is $\text{Var}(X)$?

20. **(5.20)** If 65 percent of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain

   (a) at least 50 who are in favor of the proposition;

   (b) between 60 and 70 inclusive who are in favor;

   (c) fewer than 75 in favor.

21. **(6.22)** The joint density of X and Y is

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? Justify your answer.

(b) Find the marginal density of X.

(c) Find $P(X + Y < 1)$.

22. **(6.33)** The monthly worldwide average number of airplane crashes of commercial airlines is 2.2. What is the probability that there will be

    (a) more than 2 such accidents in the next month?

    (b) more than 4 such accidents in the next 2 months?

    (c) more than 5 such accidents in the next 3 months?

    Justify your answers completely.

23. **(7.4)** If X and Y have joint density function

$$f_{X,Y}(x, y) = \begin{cases} 1/y, & 0 < y < 1,\ 0 < x < y \\ 0, & \text{otherwise,} \end{cases}$$

    find

    (a) $\mathbb{E}[XY]$

    (b) $\mathbb{E}[X]$

    (c) $\mathbb{E}[Y]$

    (d) $\operatorname{Cov}(X, Y)$.

24. If X is a Poisson random variable with parameter $\lambda > 0$, determine the moment generating function of X.

25. A fair coin is tossed 100 times. Find the expected value and the standard deviation of the number of heads. What does Chebyshev's inequality tell you about the probability that the number of heads that turn up deviates from the expected number by three standard deviations?

26. Let X be a normal random variable with mean μ and variance σ^2. Use the Chebyshev inequality to estimate the probability that X deviates from its mean by an amount more than 3σ. Use the table of values of area under the normal curve to determine the exact value for this probability.

27. A random number X is drawn repeatedly from the uniform distribution in the interval $(0, 1)$. Use the Central Limit Theorem to estimate the probability that the average value over 10,000 realizations deviates from the mean 0.5 by more than 0.01.

28. Use the Central Limit Theorem to estimate the probability that among $10,000$ random digits the digit 3 appears not more than 931 times.