Wed, May 1, 2013 **MATH 340, Midterm 2** Prof. V. Panferov

Name: (print) Solutions.

This test includes 7 questions (total of 42 points), on 7 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) The lifetime in hours of a certain electronic device is a random variable with probability density given by

$$f(x) = xe^{-x}, \quad x > 0.$$

Compute the expected lifetime of such a device.

Let X - lifetime of the device.

$$\mathbb{E}[X] = \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x \cdot x e^{-x}\,dx = \int_0^\infty x^2 e^{-x}\,dx$$

$$= \underbrace{\left[-x^2 e^{-x}\right]_0^\infty}_{=0} + 2\int_0^\infty x e^{-x}\,dx$$

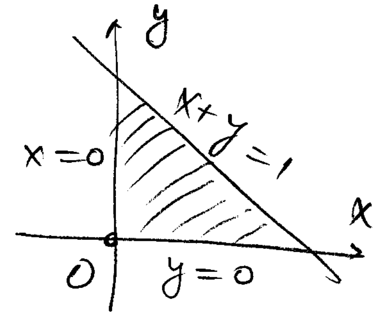
$$= \underbrace{\left[-2x e^{-x}\right]_0^\infty}_{=0} + 2\int_0^\infty e^{-x}\,dx$$

$$= \left[-2e^{-x}\right]_0^\infty = 2 \quad (\text{hours.})$$

Also, note that $\int_0^\infty x^2 e^{-x}\,dx = \Gamma(3) = 2! = 2.$

2. (6 points) The joint probability density for random variables X and Y is given by

$$f(x,y) = \begin{cases} Cxy, & x > 0,\ y > 0,\ x + y < 1 \\ 0, & \text{otherwise.} \end{cases}$$



(a) Determine the constant C.

(b) Are X and Y independent?

(a)

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} Cxy\,dy\,dx$$

$$= \int_0^1 Cx\,\frac{(1-x)^2}{2}\,dx \qquad (z = 1-x)$$

$$= \int_0^1 \frac{C}{2}(1-z)z^2\,dz$$

$$= \frac{C}{2}\int_0^1 (z^2 - z^3)\,dz = \frac{C}{2}\cdot\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{C}{24}$$

$$\Rightarrow C = 24.$$

(b)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} 24xy\,dy > 0$$

for any $0 < x < 1$.

$$f_Y(x) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} 24xy\,dx > 0$$

for any $0 < y < 1$.

$\Rightarrow f_X(x) f_Y(y) > 0$ everywhere on $x, y: 0 < x < 1,\ 0 < y < 1$ $\Rightarrow f_X(x) f_Y(y) \neq f(x,y)$ $\Rightarrow$ not independent.

Continued...

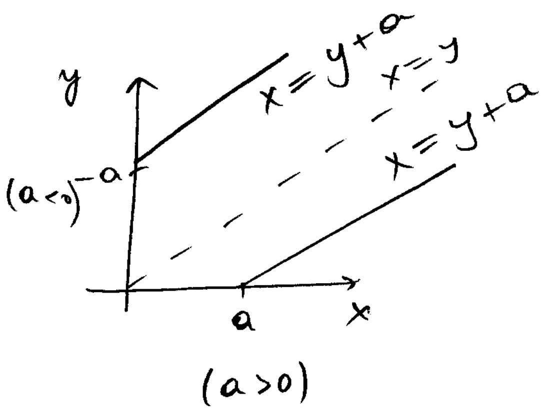
3. (6 points) Let X and Y be independent, exponentially distributed random variables, with parameter $\lambda = 1$. Find the probability density of $X - Y$.

$$X \sim \text{Exponential}(1), \quad Y \sim \text{Exponential}(1)$$

$$f_X(x) = e^{-x}, \quad x > 0, \quad f_Y(y) = e^{-y}, \quad y > 0.$$

$$F_{X-Y}(a) = P(X - Y \le a) = P(X \le Y + a)$$

$$\overset{(a>0)}{=} \int_0^\infty \int_0^{y+a} e^{-x} e^{-y}\, dx\, dy$$



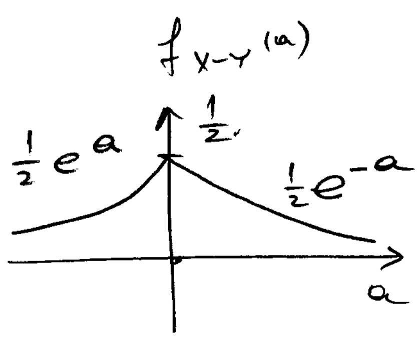
$$= \int_0^\infty e^{-y}\left(1 - e^{-(y+a)}\right) dy$$

$$= \int_0^\infty e^{-y} dy - e^{-a} \int_0^\infty e^{-2y}\, dy$$

$$= 1 - \frac{1}{2} e^{-a};$$

$$\overset{(a<0)}{=} \int_{-a}^\infty \int_0^{y+a} e^{-x} e^{-y}\, dx\, dy$$

$$= \int_{-a}^\infty e^{-y}\left(1 - e^{-(y+a)}\right) dy$$



$$= \int_{-a}^\infty e^{-y} dy - e^{-a} \int_{-a}^\infty e^{-2y}\, dy$$

$$= e^{a} - e^{-a} \frac{1}{2} e^{2a} = \frac{1}{2} e^{a}$$

$$f_{X-Y}(a) = F'_{X-Y}(a) = \begin{cases} \frac{1}{2} e^{-a}, & a > 0 \\ \frac{1}{2} e^{a}, & a < 0 \end{cases} = \frac{1}{2} e^{-|a|}.$$

Continued...

4. (6 points) The gross weekly sales at a certain store is a normal random variable with mean \$2,700 and the standard deviation \$300. What is the probability that

   (a) the total gross sales over the next two weeks will be less than \$5,000?

   (b) weekly sales exceed \$3,000 in at least two of the next three weeks?

   State the independence assumptions that you are making.

$$X_i \sim \text{Normal}(\mu = 2700, \sigma^2 = 300^2).$$

(a) $$P(X_1 + X_2 < 5{,}000) = P\left(\frac{X_1+X_2-5400}{\sqrt{2}\cdot 300} < \frac{5000-5400}{\sqrt{2}\cdot 300}\right)$$

Assume X_1, X_2 – sales in the two weeks are independent.

$$= P\left(Z < -\frac{400}{\sqrt{2}\cdot 300}\right)$$

$$\Rightarrow X_1 + X_2 \sim \text{Normal}(\mu = 5400, \sigma^2 = 2 \cdot 300^2)$$

$$= P\left(Z < -\frac{4}{3\sqrt{2}}\right)$$

$$= P\left(Z > \frac{4}{3\sqrt{2}}\right) = 1 - \Phi(0.9428\ldots)$$

$$\approx 1 - 0.8264 = 0.1736.$$

(b) $$P(X_i > 3000) = P\left(\frac{X_i - 2700}{300} > \frac{3000-2700}{300}\right)$$

$$= P(Z > 1) = 1 - P(Z \le 1) = 1 - \Phi(1)$$

$$= 1 - 0.8413 = 0.1587 = p.$$

Let N – number of weeks with $X_i > 3000$.

$$P(N \geq 2) = P(N=2) + P(N=3)$$

$$= \binom{3}{2} p^2 (1-p)^1 + p^3 = 0.06756\ldots$$

Continued...

5. (6 points) A man and a woman agree to meet at a certain location at about 12 : 30 pm. The man arrives at a time uniformly distributed between 12 : 20 and 12 : 40 and if the woman independently arrives at a time uniformly distributed between 12 : 15 and 12 : 45 pm.

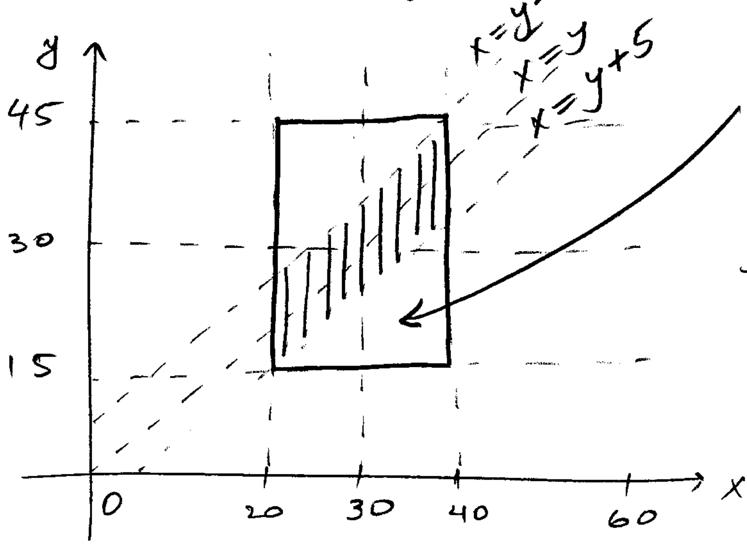
   (a) Find the probability that the first to arrive waits no longer than 5 minutes.

   (b) What is the probability that the man arrives first?

(a) Let $X \in (20, 40)$ – arrival time of the man
$Y \in (15, 45)$ – arrival time of the woman.

Then
$$f(x,y) = \begin{cases} \frac{1}{600}, & 20 < x < 40,\ 15 < y < 45 \\ 0, & \text{otherwise} \end{cases}$$

is the joint density.



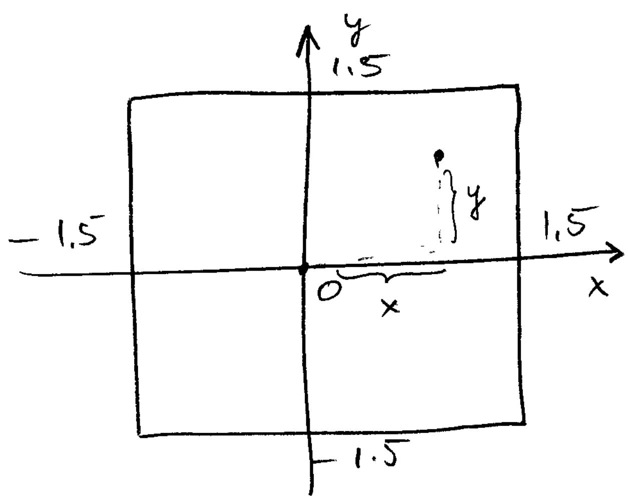
$f(x,y) = \frac{1}{600}$ here

shaded region – $x, y: \ |x-y| \le 5$

$$P(|X-Y| \le 5) = \frac{20 \cdot 10}{600} = \frac{1}{3}$$

(b) $P(X < Y) = \frac{1}{2}$. (area above $x=y$ relative to the area of the rectangle)

6. (6 points) The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are $(0,0)$ to the point (x,y) is $|x|+|y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.



Joint density: $f(x,y) = \begin{cases} \frac{1}{9}, & -1.5 < x < 1.5,\ -1.5 < y < 1.5 \\ 0, & \text{otherwise} \end{cases}$

$$E[|X|+|Y|] = \int_{-1.5}^{1.5}\int_{-1.5}^{1.5} \frac{1}{9}(|x|+|y|)\,dx\,dy$$

$$= \frac{1}{3}\int_{-1.5}^{1.5} |x|\,dx + \frac{1}{3}\int_{-1.5}^{1.5} |y|\,dy$$

$$= \frac{2}{3}\int_{-1.5}^{1.5} |x|\,dx = \frac{4}{3}\int_{0}^{1.5} x\,dx$$

$$= \frac{4}{3}\frac{(1.5)^2}{2} = \frac{2}{3}\cdot\left(\frac{3}{2}\right)^2 = \frac{3}{2}.$$

Continued...

7. (6 points) Let X be uniformly distributed over $(0,1)$. Find the density of the random variable $Y = -\frac{1}{\lambda}\log X$, where λ is a positive constant. (Here notation log is used for the logarithm to base e).

$$P(Y \le a) = P\left(-\frac{1}{\lambda}\log X \le a\right) = P(\log X \ge -\lambda a)$$

$$= P(X \ge e^{-\lambda a}) = 1 - e^{-\lambda a},$$

$a > 0$.

$$f_Y(a) = \frac{d}{da} P(Y \le a) = \lambda e^{-\lambda a}, \quad a > 0$$

$\Rightarrow Y \sim \text{Exponential}(\lambda)$.

The end.