Wed, Mar 13, 2013 **MATH 340, Midterm 1** Prof. V. Panferov

Name: (print) Solutions.

This test includes 9 questions (total of 50 points), on 8 pages. The duration of the test is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	9	total

Important: The test is closed books/notes. A basic scientific calculator is allowed; no graphing calculators or other electronic devices. Give complete solutions to problems; no credit will be given for just the correct answers.

1. (6 points) (a) How many different letter arrangements are possible from the word $PEPPER$?

$P_1 E_1 P_2 P_3 E_2 R \rightarrow 6!$ different arrangements.

identify $P_1 \leftrightarrow P_2 \leftrightarrow P_3$ (3! permutations)

and $E_1 \leftrightarrow E_2$ (2! permutations)

$$\Rightarrow \frac{6!}{2!3!} = \frac{1\cdot\cancel{2}\cdot\cancel{3}\cdot 4\cdot 5\cdot \cancel{6}^{3}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot 1\cdot\cancel{2}} = 3\cdot 4\cdot 5 = 60$$ unique arrangements.

(b) If a letter is selected at random from the word $PEPPER$, and then another one at random from the word $VERTICAL$, what is the probability that the same letter is selected?

$$\{\text{Same letter selected}\} = \underbrace{\{E \text{ is selected}\} \cup \{R \text{ is selected}\}}_{\text{mutually exclusive.}}$$

$$P\{\text{same letter selected}\} = \frac{2\cdot 1}{6\cdot 8} + \frac{1\cdot 1}{6\cdot 8} = \frac{3}{48} = \frac{1}{16}.$$

2. (6 points) (a) If $P(A) = 0.7$ and $P(B) = 0.9$, show that $P(AB) \geq 0.6$.

$$P(A^c) = 0.3 \;; \qquad P(B^c) = 0.1$$

$$\underset{\parallel}{P(A^c \cup B^c)} \leq P(A^c) + P(B^c) = 0.4$$

$$P((AB)^c) = 1 - P(AB) \leq 0.4$$

$$\Rightarrow P(AB) \geq 1 - 0.4 = 0.6$$

(b) In general, use axioms of probability to prove Bonferroni's inequality:

$$P(AB) \geq P(A) + P(B) - 1.$$

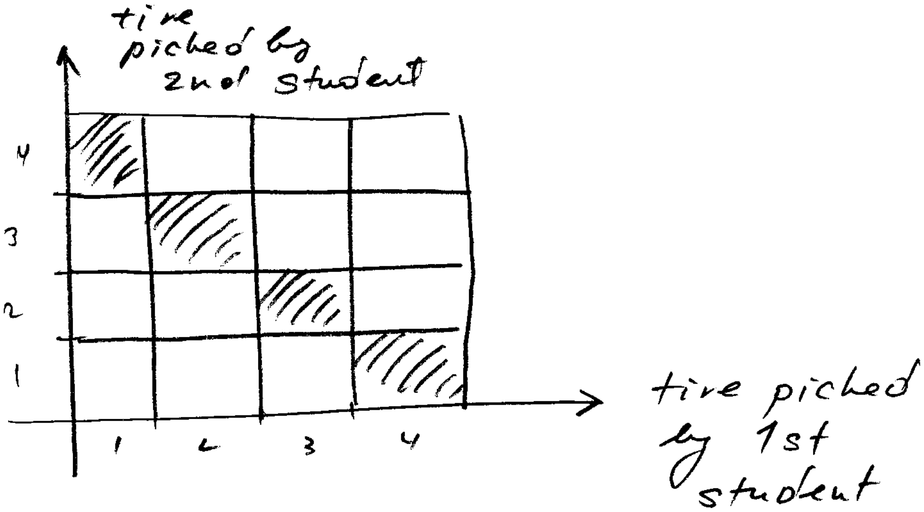
$$P(AB) = 1 - P((AB)^c) = 1 - P(A^c \cup B^c)$$

$$\geq 1 - P(A^c) - P(B^c) =$$

$$= 1 - (1 - P(A)) - (1 - P(B))$$

$$= P(A) + P(B) - 1.$$

3. (6 points) Two college students arrive late for the math final exam and give the professor a phony excuse that their car had a flat tire. Well, the professor says, "each of you write down on a piece of paper which of the four tires was flat". What is the probability that both students pick the same tire?



▨ - same tire picked.

$$P\left(\text{they pick the same tire}\right) = \frac{\#(\text{of outcomes in } E)}{\text{total \# of outcomes}}$$

$$= \frac{4}{16} = \frac{1}{4}.$$

Continued...

4. (6 points) There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the coins is selected at random and flipped, it shows heads. What is the probability that it was a two-headed coin?

E_i – coin $\# i$ was picked; H – heads

$P(H/E_1) = 1$; $P(H/E_2) = \frac{1}{2}$; $P(H/E_3) = \frac{3}{4}$.

$$P(E_1/H) = \frac{P(E_1 H)}{P(H)} = \frac{P(E_1 H)}{P(HE_1) + P(HE_2) + P(HE_3)}$$

$$= \frac{P(H/E_1)P(E_1)}{P(H/E_1)P(E_1) + P(H/E_2)P(E_2) + P(H/E_3)P(E_3)}$$

$$= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3}} = \frac{1}{1 + \frac{1}{2} + \frac{3}{4}}$$

$$= \frac{4}{4 + 2 + 3} = \frac{4}{9}.$$

5. (6 points) (a) Define the probability mass function for the Poisson random variable with parameter λ.

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2\ldots$$

(b) Derive a formula for $\mathbb{E}[X]$ if X is Poisson distributed with parameter λ.

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\, P(X=k) = \sum_{k=1}^{\infty} k\, P(X=k)$$

$$= \sum_{k=1}^{\infty} k\, e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^k}{(k-1)!}$$

$$= e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^{j+1}}{j!} = \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

Continued...

6. (6 points) The probability of being dealt a full house in a hand of poker is approximately 0.0014. Find an approximation for the probability that, in 1000 hands of poker, you will be dealt at least 2 full houses.

Exact: $X \sim \text{Binomial}(n=1000,\ p=0.0014)$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$= 1 - p^0 \cdot (1-p)^{1000} - \binom{1000}{1} p^1 (1-p)^{999}$$
$$= 1 - (0.9986)^{1000} - 1.4\,(0.9986)^{999}$$
$$= 0.4083\ldots$$

Approximate: $X \sim \text{Poisson}(\lambda = np = 1.4)$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$
$$= 1 - e^{-1.4} - 1.4e^{-1.4}$$
$$= 0.4082\ldots$$

7. (6 points) Suppose that the cumulative distribution function of X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x/8, & 0 \le x < 1 \\ 1/4 + (x-1)/4, & 1 \le x < 2 \\ 7/8, & 2 \le x < 4 \\ 1, & x \ge 4. \end{cases}$$

(a) Find $P(X = k)$, $k = 0, 1, 2, 3, 4$.

(b) Find $P(0.5 < X \le 2.5)$.

(a) $P(X=0) = 0$ ($F(x)$ is continuous at 0.)

$P(X=1) = F(1) - F(1-0) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$

$P(X=2) = F(2) - F(2-0) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$

$P(X=3) = 0$ ($F(x)$ is continuous at 3)

$P(X=4) = F(4) - F(4-0) = 1 - \frac{7}{8} = \frac{1}{8}$.

(b) $P(0.5 < X \le 2.5) = F(2.5) - F(0.5) =$

$= \frac{7}{8} - \frac{1}{16} = \frac{13}{16}$.

8. (4 points) There are n socks, 3 of which are red, in a drawer. What is the value of n if, when 2 socks are chosen randomly, the probability that they are both red is $\frac{1}{2}$?

$$P(\text{both red}) = \frac{\binom{3}{2}}{\binom{n}{2}} = \frac{1}{2}$$

$$\frac{1\cdot 2\cdot 3}{1\cdot 2} = \frac{1}{2}\cdot\frac{n(n-1)}{1\cdot 2}$$

$$n(n-1) = 12$$

$$n = 4 \text{ or } \cancel{n = -3}$$ ← n can't be negative

$$\Rightarrow n = 4.$$

9. (4 points) Suppose that $P(X=0) = 1 - P(X=1)$. If $\mathbb{E}[X] = 5\mathrm{Var}(X)$, find $P(X=1)$.

$$P(X=1) = p \Rightarrow P(X=0) = 1-p$$

$$\mathbb{E}[X] = p = 5\,\mathrm{Var}(X) = 5p(1-p)$$

$$\Rightarrow p = 0 \text{ or }$$

$$1 = 5(1-p)$$

$$1-p = \frac{1}{5}$$

$$p = \frac{4}{5}$$

Answer: $p = 0$ or $p = \frac{4}{5}$.

The end.