Apr 21, 2014

MATH 340: Midterm 2 Review Questions

1. Give an example of two distinct discrete random variables with the same probability mass function.

2. A random variable X has probability density

$$f(x) = \frac{a}{x^2+1}, \quad -\infty < x < \infty.$$

   Find

   (a) The constant a.

   (b) The cumulative distribution function of X.

   (c) The probability $P(|X| < 1)$.

   Answers: (a) $1/\pi$; (b) $\frac{1}{2} + \frac{1}{\pi}\arctan x$; (c) $1/2$.

3. A random variable X has cumulative distribution function

$$\Phi(a) = a + b \arctan\frac{x}{2}, \quad -\infty < x < \infty.$$

   (a) Find the constants a and b. (b) Find the probability density function of X.

4. Balls are drawn from an urn containing w white balls and b black balls unitil a white ball appears. Find the expected value μ and the variance σ^2 of the number of black balls drawn, assuming that each ball is replaced after being drawn.

   Answer: $\mu = b/w$, $\sigma^2 = b(w+b)/w^2$.

5. A textbook of 500 pages contains 500 misprints. Estimate the probability that a given page contains exactly 3 misprints.

6. Suppose that $P(X = 0) = 1 - P(X = 1)$. If $\mathbb{E}[X] = 5\text{Var}(X)$, find $P(X = 1)$.

7. A point X is picked at random (with uniform density) in the interval $(0, 1)$. Find the probability that $X > 1/2$, given that

   (a) $X > 1/4$

   (b) $|X - 1/2| < 1/4$.

8. Suppose that the cumulative distribution function of X is given by

$$F(a) = \begin{cases} 0, & a < 0 \\ 1/8, & 0 \le a < 1 \\ 1/4, & 1 \le a < 2 \\ 7/8, & 2 \le a < 4 \\ 1, & a \ge 4. \end{cases}$$

   (a) Find $P(X = i)$ for $i = 0, 1, 2, 3, 4$.

   (b) Find $\mathbb{E}[X]$.

9. The cumulative distribution function of Z is given by

$$F_Z(z) = \begin{cases} 0, & z < a \\ (z-a)/(b-a), & a \le z \le b \\ 1, & z > b, \end{cases}$$

where a and b are given numbers.

   (a) Find the probability density of X.
   (b) Find $\mathbb{E}[X]$.
   (c) Find $\text{Var}(X)$.

10. Find the mean and the variance of the random variable X with probability density

$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty.$$

Answer: $\mathbb{E}[X] = 0$, $\text{Var}(X) = 2$.

11. Let X be an exponential random variable. Show that $P(X > s+t|X > t) = P(X > s)$ for $s > 0$, $t > 0$ (the "memoryless property").

12. Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is the probability that the repair time exceeds 4 hours? If it exceeds 4 hours what is the probability that it exceeds 8 hours?

13. Let X be random variable normally distributed with parameters $\mu = 70$, $\sigma = 10$. Estimate

   (a) $P(X > 50)$
   (b) $P(X \le 60)$
   (c) $P(|X - 60| > 20)$.

14. An electronic component has lifetime X which is distributed according to the probability density

$$f(x) = cx^2e^{-x}, \quad x > 0,$$

where c is a constant.

   (a) Determine the value of the constant c.
   (b) Compute the expected lifetime of the device.

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15. **(5.18)** Suppose that X is a normal random variable with mean 5. If $P(X > 9) = 0.2$, approximately what is $\text{Var}(X)$?

16. **(5.20)** If 65 percent of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain

(a) at least 50 who are in favor of the proposition;

(b) between 60 and 70 inclusive who are in favor;

(c) fewer than 75 in favor.

17. Suppose you toss a dart at a circular target of radius 10 inches. Given that the dart lands in the upper half of the target, find the probability that

    (a) it lands in the right half of the target

    (b) its distance from the center is less than 5 inches

    (c) its distance from the center is more than 5 inches

    (d) it lands within 5 inches of the point $(0, 5)$ (the origin is the center of the circle).

18. A man and a woman agree to meet at a certain location at about $12:30$ pm. The man arrives at a time uniformly distributed between $12:20$ and $12:40$ and if the woman independently arrives at a time uniformly distributed between $12:15$ and $12:45$ pm.

    (a) Find the probability that the first to arrive waits no longer than 5 minutes.

    (b) What is the probability that the man arrives first?

    (See related example in Section 6.2.)

19. **(6.22)** The joint density of X and Y is

$$f(x,y) = \begin{cases} x+y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

    (a) Are X and Y independent? Justify your answer.

    (b) Find the marginal density of X.

    (c) Find $P(X + Y < 1)$.

Multiple Integration Review Questions

(mostly from J. Stewart, Calculus, 7th ed. Sections 15.3, 15.4)

1. Sketch the region of integration and and change the order of integration:

   (a) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$

   (b) $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$

   (c) $\int_0^\infty \int_x^\infty f(x,y)\,dy\,dx.$

2. Evaluate the integral by changing the order of integration:

   (a) $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

   (b) $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$

   (c) $\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx.$

3. Use geometry or symmetry, or both, to evaluate the double integral:

   (a) $\iint_D (x+2)\,dx\,dy, \quad D = \{(x,y) : 0 \le y \le \sqrt{9-x^2}\}$

   (b) $\iint_D ax^3 + by^3 + \sqrt{a^2-x^2}\;dx\,dy, \quad D = [-a,a] \times [-b,b].$

4. Evaluate the integrals by converting to polar coordinates:

   (a) $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$

   (b) $\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy.$

5. Use polar coordinates to evaluate the integral:

   $$\iint_D e^{-x^2-y^2}\,dx\,dy, \quad \text{where } D \text{ is the region bounded by}$$

   the cemicircle $x = \sqrt{4-y^2}$ and the y-axis.