Wed, Nov 18, 2009 **MATH 262** **Midterm 2**

Name: (print) ______________________

CSUN ID No. : Solutions

This exam includes 9 questions, the last question is a bonus. Please check that your copy has 10 pages. The duration of the exam is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	9	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your work.

1. (8 points) For the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Find a basis of $\ker(A)$ and determine its dimension.

Solution of the homogeneous linear system:

$$\begin{aligned} x_1 &= 2u+t \\ x_2 &= u \\ x_3 &= -5t \\ x_4 &= t \\ x_5 &= 0 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\ker(A) = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{linearly independent}} \right\}$$

("1" and "0" in the components 2 and 4)

Basis: $\begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; Dimension of the kernel $= 2$

(b) Find all redundant vectors among the vector columns of the matrix A and write them as linear combinations of the remaining vectors.

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \ker(A) \Rightarrow \text{ linear relation } 2v_1 + v_2 + 0 \cdot v_3 + 0 \cdot v_4 + 0 \cdot v_5 = 0$$

$$\Rightarrow v_2 = -2v_1 \text{ - redundant}$$

$$\begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \in \ker(A) \Rightarrow \text{ linear relation } v_1 + 0v_2 - 5v_3 + v_4 + 0 \cdot v_5 = 0$$

$$\Rightarrow v_4 = 5v_3 - v_1 \text{ — redundant}$$

(c) Find a basis of $\operatorname{im}(A)$ and determine its dimension.

$$\operatorname{im}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

excluding redundant vectors,

$$= \operatorname{span}\underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\text{lin. independent, basis;}} = \mathbb{R}^3$$

$$\dim(\operatorname{im}(A)) = 3.$$

2. (8 points) Find a basis of $W^\perp$ where $W = \text{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}$

Find ker of $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\ker(A) = \text{span}\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}\right\}$$

We have $W^\perp = \ker(A)$

$$\left(x \in W^\perp \iff x \cdot \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 0,\ x \cdot \begin{pmatrix}1\\2\\3\\4\end{pmatrix} = 0 \right.$$

$$\iff \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} x = 0$$

$$\left. \iff x \in \ker(A) \right)$$

$\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$ are linearly independent, and span $\ker(A)$

$\Rightarrow$ they form a basis.

3. (8 points) Consider the vectors

$$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

Find a vector u_3 so that the vectors u_1, u_2, u_3 form an orthonormal basis of $\mathbb{R}^3$. How many possible vectors u_3 are there?

First, find a vector orthogonal to $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}$

(By guessing, or by solving the kernel of

$$A = \begin{pmatrix}1 & 1 & 1\\ -1 & 1 & 0\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{2}\end{pmatrix}$$

$$\ker(A) = \operatorname{span}\left\{\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\}.)$$

$$V_3 = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

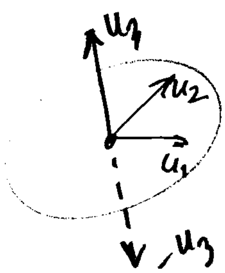
Then, normalize: $\|v_3\| = \sqrt{4+1+1} = \sqrt{6}$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

There are two possible choices for u_3:

$$\pm\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}.$$

(two possible unit vectors perpendicular to a given plane)



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4. (8 points) Consider the set Q of all 2×2 matrices A that commute with $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (i. e. satisfy $AB = BA$).

(a) Verify that Q is a subspace of $\mathbb{R}^{2\times 2}$.

$$\begin{array}{rl} & A_1 B = BA_1 \\ + & A_2 B = BA_2 \\ \hline & (A_1+A_2)B = B(A_1+A_2) \end{array} \Rightarrow \text{sum property OK.}$$

$$\begin{array}{l} A_1 B = BA_1 \quad \cdot k \\ (kA_1)B = B(kA_1) \end{array} \Rightarrow \begin{array}{l}\text{scalar multiple} \\ \text{property OK}\end{array}$$

$\Rightarrow$ subspace.

(b) Find a basis of Q and determine its dimension.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix}$$

$$\Leftrightarrow \begin{array}{l} a+b = a+c \\ c+d = a+c \\ a+b = b+d \end{array} \Leftrightarrow \begin{array}{l} b = c \\ d = a \end{array}$$

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Basis: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Dimension $= 2$.

5. Which of the following are subspaces of the space P_2 of all polynomials $p(x) = a + bx + cx^2$? Explain your answer.

   (a) All polynomials $p(x)$ of degree 2.

   (b) All polynomials $p(x)$ such that $p(1) = p(2)$.

   (c) All polynomials $p(x)$ such that $p(0) = 1$.

   (d) All polynomials $p(x)$ such that $p(1) = 0$.

(a) No. zero polynomial is not of degree 2

(b) Yes. $p(1) = a+b+c$
$p(2) = a+2b+4c$

$a+b+c = a+2b+4c \Leftrightarrow b+3c=0$

$p(x) = a - 3cx + cx^2 = a \cdot 1 + c(-3x+x^2)$.

The set $=$ span $\{1, -3x+x^2\}$ – subspace

(c) No. zero polynomial does not satisfy that.

(d) Yes.

Addition OK:

$p(1)=0$
$q(1)=0$ $\Rightarrow p(1)+q(1)=0$

Multiplication by scalar OK:

$p(1)=0 \Rightarrow kp(1)=0$.

6. (8 points) (a) Determine if the vector x is in the linear span of the vectors v_1, v_2:

$$x = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix},$$

$$A = \begin{pmatrix} -1 & 1 & -3 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 3 \\ 0 & 4 & -4 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\ker(A) = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$; $\Rightarrow$ linear relation

$-2X + V_1 + V_2 = 0$

$$X = \frac{1}{2} V_1 + \frac{1}{2} V_2$$

(b) If it is, find the coordinates of x in the basis (v_1, v_2).

$$X \in \operatorname{span}\{V_1, V_2\}$$

$$C_X = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

7. (8 points) Find the matrix B of the linear transformation $T(x) = Ax$ with respect to the basis (v_1, v_2, v_3)

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Compute $Av_1 = \begin{pmatrix} 18 \\ 9 \\ -18 \end{pmatrix} = 9v_1$

$Av_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad Av_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{array}{ccc} x = c_1 v_1 + c_2 v_2 + c_3 v_3 & \xrightarrow{A} & \begin{aligned} y &= Ax \\ &= c_1 Av_1 + c_2 Av_2 + c_3 Av_3 \\ &= 9c_1 v_1 \end{aligned} \\ \uparrow & & \\ c_x = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} & \xrightarrow{B} & c_y = \begin{pmatrix} 9c_1 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$$B = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

8. (8 points) State which of the following statements are true or false. (You do not need to show your work.)

   (a) If A is a 5×6 matrix (5 rows, 6 columns) of rank 4 then the kernel of A has dimension 1.

   (b) If A is a matrix with linearly independent columns then the kernel consists of the zero vector only.

   (c) If the image of an $n \times n$ matrix A is all of $\mathbb{R}^n$ then A must be invertible.

   (d) If a subspace V is spanned by two vectors then any three vectors in V must be linearly dependent.

   (e) The lower triangular 2×2 matrices form a subspace of $\mathbb{R}^{2\times 2}$.

   (f) There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$. [$V^\perp$ denotes the orthogonal complement.]

   (g) The kernel of a linear transformation is a subspace of the domain.

   (h) If the vectors u_1, u_2, u_3 in $\mathbb{R}^3$ are orthonormal then they must be linearly independent.

Answers:

(a) false $\quad$ ($\dim(\ker(A)) = 2$)

(b) true $\quad$ (definition of linear independence)

(c) true $\quad$ ($\ker(A) = 0$ then, by the rank-nullity thm.)

(d) true

(e) true

(f) false $\quad$ ($\dim(V^\perp) = \dim(\ker(A))$ $\Rightarrow \dim(V^\perp) + \dim(V) = 5$ by rank-nullity theorem)

(g) true

(h) true

9. (bonus: 8 points) Are the following matrices similar? Explain your answer.

(a) $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. No.

Rotation through $\vartheta = \frac{\pi}{4}$

Rotation through $\vartheta = \frac{\pi}{2}$

There is no system of coordinates $x = Sc_x$ such that

$$y = Ax \iff c_y = Bc_x.$$

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$SB = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$; $AS = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$

$SB = AS \Rightarrow c = 0,\ d = 0,\ b = 0 \Rightarrow S = 0.$

$\Rightarrow$ There is no invertible matrix S such that

$$SB = AS.$$