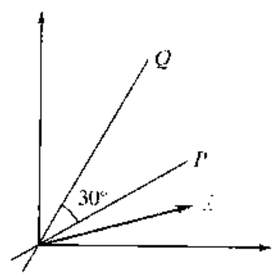
a. For two angles, α and β, consider the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Arguing geometrically, describe the linear transformations $\vec{y} = D_\alpha D_\beta \vec{x}$ and $\vec{y} = D_\beta D_\alpha \vec{x}$. Are the two transformations the same?

b. Now compute the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Do the results make sense in terms of your answer in part (a)? Recall the trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

30. Consider the lines P and Q in $\mathbb{R}^2$ sketched below. Consider the linear transformation $T(\vec{x}) = \text{ref}_Q(\text{ref}_P(\vec{x}))$, that is, we first reflect $\vec{x}$ about P and then we reflect the result about Q.



a. For the vector $\vec{x}$ given in the figure, sketch $T(\vec{x})$. What angle do the vectors $\vec{x}$ and $T(\vec{x})$ enclose? What is the relationship between the lengths of $\vec{x}$ and $T(\vec{x})$?

b. Use your answer in part (a) to describe the transformation T geometrically, as a reflection, rotation, shear, or projection.

c. Find the matrix of T.

d. Give a geometrical interpretation of the linear transformation $L(\vec{x}) = \text{ref}_P(\text{ref}_Q(\vec{x}))$, and find the matrix of L.

31. Consider two matrices A and B whose product AB is defined. Describe the ith row of the product AB in terms of the rows of A and the matrix B.

32. Find all 2×2 matrices X such that $AX = XA$ for all 2×2 matrices A.

For the matrices A in Exercises 33 through 42, compute $A^2 = AA$, $A^3 = AAA$, and A^4. Describe the pattern that emerges, and use this pattern to find $A^{1,001}$. Interpret your answers geometrically, in terms of rotations, reflections, shears, and orthogonal projections.

33. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **35.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ **38.** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

39. $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ **40.** $\dfrac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

41. $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ **42.** $\dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

In Exercises 43 through 48, find a 2×2 matrix A with the given properties. (Hint: It helps to think of geometrical examples.)

43. $A \neq I_2$, $A^2 = I_2$ **44.** $A^2 \neq I_2$, $A^4 = I_2$

45. $A^2 \neq I_2$, $A^3 = I_2$

46. $A^2 = A$, all entries of A are nonzero.

47. $A^3 = A$, all entries of A are nonzero.

48. $A^{10} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

In Exercises 49 through 54, consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Compute the indicated products. Interpret these products geometrically, and draw composition diagrams, as in Example 2.

49. AF and FA **50.** CG and CG

51. FJ and JF **52.** JH and HJ

53. CD and DC **54.** BE and EB.

In Exercises 55 through 64, find all matrices X that satisfy the given matrix equation.

55. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

56. $X \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **57.** $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} X = I_2$

58. $X \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = I_2$ **59.** $X \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = I_2$

60. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = I_2$ **61.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} X = I_2$

62. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} X = I_2$ **63.** $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} X = I_3$

64. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} X = I_3$

65. Find all upper triangular 2×2 matrices X such that X^2 is the zero matrix.