Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. Find the basis of the kernel of the matrices

(a) $\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$

$x_1 = -\frac{3}{2}t$

$x_2 = t$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} t$

$\ker\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}\right\}$

$= \operatorname{span}\left\{\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right\}$

Basis: $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

(b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 2 free variables;

$x_1 = u$

$x_2 = t$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = u\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

$= \mathbb{R}^2$

Basis: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2. Consider a matrix A and let $B = \text{rref}(A)$.

(a) Is $\ker(A)$ necessarily equal to $\ker(B)$? Explain.

Yes. Linear systems $Ax=0$ and $Bx=0$ have the same solutions, since one is obtained from the other by row reduction.

(b) Is $\text{im}(A)$ necessarily equal to $\text{im}(B)$? Explain.

No. Row reduction changes the column space.

Example: $\operatorname{Im}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\operatorname{Im}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Please turn over...

3. Consider three linearly independent vectors v_1, v_2, v_3 in $\mathbb{R}^n$. Are the vectors v_1, v_1+v_2, $v_1+v_2+v_3$ linearly independent as well? Explain your answer.

Yes.

$$c_1 v_1 + c_2 (v_1+v_2) + c_3 (v_1+v_2+v_3) = 0$$

$$(c_1+c_2+c_3) v_1 + (c_2+c_3) v_2 + c_3 v_3 = 0$$

$$\Rightarrow \quad \begin{aligned} c_3 &= 0 \\ c_2+c_3 &= 0 \\ c_1+c_2+c_3 &= 0 \end{aligned} \qquad \Rightarrow \quad \begin{aligned} c_3 &= 0 \\ c_2 &= 0 \\ c_1 &= 0 \end{aligned}$$

$\Rightarrow$ linearly independent

4. Determine whether the following vectors form a basis of $\mathbb{R}^4$:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

all columns have pivot elements

$\Rightarrow$ columns are lin. independent

$\Rightarrow$ form a basis of $\mathbb{R}^4$.