Mon, Oct 12, 2009 **MATH 262** **Midterm 1**

Name: (print) ____________________

CSUN ID No. : Solutions.

This exam includes 9 questions, on 8 pages. The duration of the exam is 1 hour 15 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	8	9	total

Note: Write solutions in the space provided. If you run out of space you may use the back of the page. Show all your steps. The test is closed books/notes. Graphing calculators are not allowed.

1. (4 points) Compute the matrix products if they are defined

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{pmatrix} a+c & b+d \\ c & d \\ 0 & 0 \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}$

not defined: # of columns of the 1st factor $\neq$ # of rows in the 2nd one.

2. (6 points) Find the rank of the matrices

(a) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & 17 & 8 \\ 0 & 9 & 33 \\ 0 & 0 & 11 \end{pmatrix}$

rearrange rows to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(this does not change rank)

$\Rightarrow$ rank $= 4$

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

rank $= 2$

upper triangular, nonzero entries on the diagonal

$\Rightarrow$ rank $= 3$

(reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
)

3. (6 points) (a) Write the system

$$\begin{cases} x + 2y + 3z = 1 \\ 4x + 5y + 6z = 5 \\ 7x + 8y + 9z = 9 \end{cases}$$

in matrix form $A\mathbf{x} = \mathbf{b}$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$$

(b) Find all solutions of the linear system in (a).

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 1 \\ 0 & -6 & -12 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 5/3 \\ 0 & 1 & 2 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = \frac{5}{3} + t \\ y = -\frac{1}{3} - 2t \\ z = t \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t$$

Continued...

4. (6 points) Consider the linear system

$$\begin{cases} y + 2z = 0 \\ x + 2y + 4z = 0 \\ kx + 2z = 1 \end{cases}$$

(a) Bring the system to the reduced row echelon form.

$$\cdot k, - \left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 2 & 4 & 0 \\ k & 0 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2k & 2-4k & 1 \end{array}\right) \begin{array}{l} \cdot 2, - \\ \cdot k, - \end{array}$$

$$- \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

(b) For which values of k does the system have a unique solution? No solution? Infinitely many solutions?

k is eliminated from the reduced row echelon form.
The system has a <u>unique solution</u>

$$\begin{cases} x = 0 \\ y = -1 \\ z = \frac{1}{2} \end{cases}$$

for any value of k.

5. (6 points) Consider the upper-triangular 3×3 matrix

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

(a) For which values of a, b, c is A invertible?

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \cdot \frac{e}{f}, - \quad \cdot \frac{c}{f}, - \quad \rightarrow \begin{pmatrix} a & b & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{b}{d}$$

If $f=0 \Rightarrow$ zero row $\Rightarrow$ not invertible. otherwise divide last row by f:

If $d=0 \Rightarrow$ zero row $\Rightarrow$ not invertible. otherwise divide 2^{nd} row by d!

$$\rightarrow \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \div a \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) If an upper triangular matrix is invertible is its inverse an upper triangular matrix as well?

If $a=0 \Rightarrow$ zero row $\Rightarrow$ not invertible. otherwise divide first row by a.

If $a \neq 0$, $d \neq 0$, $f \neq 0$ the matrix reduces to the identity matrix $\Rightarrow$ invertible. If any of $a, d, f = 0$ then the matrix is not invertible.

Reduction is performed by subtracting lower rows from the upper ones $\Rightarrow$ the inverse matrix will also be upper triangular.

Continued...

6. (6 points) Consider the matrices

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \qquad F = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

Classify the matrices by type of the linear transformation that they represent and fill in the blanks in the following sentences.

Matrix __E__ represents a scaling — scaling by a factor of 4.

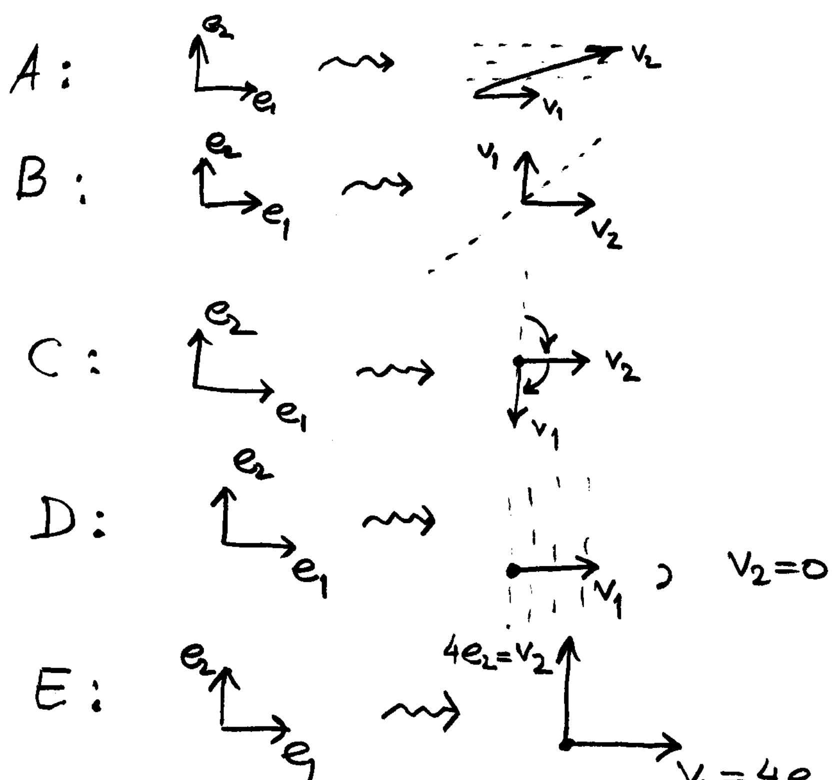
Matrix __C__ represents a rotation — rotation by 90° clockwise

Matrix __D__ represents a projection — projection on the x-axis

Matrix __B__ represents a reflection — reflection about $x=y$

Matrix __A__ represents a shear — horizontal shear $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+3y \\ y \end{pmatrix}$

Matrix __F__ is none of the above — projection onto $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ combined with a scaling.



7. (3 points) Find the image of the linear transformation with the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 1 \\ -1 & -3 & 0 \end{pmatrix}$$

Describe the image geometrically as a line, plane, etc.

$$\text{im}(A) = \text{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{= v_1}, \underbrace{\begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}}_{= v_2}, \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{= v_3} \right\}$$

By inspection, $v_2 = 3v_1$ – redundant

$$\text{im}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2 linearly independent vectors span a plane in $\mathbb{R}^3$.

8. (3 points) Find the kernel of the matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \end{pmatrix}$. Describe the kernel geometrically as a line, plane, etc.

Reduce: $\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$

$x_1 = -5t$
$x_2 = t$
$x_3 = t$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

(It is also easy to guess $\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ as the vector orthogonal to both rows of the original matrix.)

$$\ker(A) = \text{span}\left\{ \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

– line in $\mathbb{R}^3$.

9. (6 points) State which of the following statements are true or false.

   (a) The matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is in reduced row-echelon form (rref).

   (b) The kernel of a 5×4 matrix (5 rows, 4 columns) is a subspace of $\mathbb{R}^4$.

   (c) There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

   (d) The formula $AB = BA$ holds for all 2×2 matrices A and B.

   (e) $\text{rank} \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = 2$.

   (f) If a matrix A is invertible then $\ker(A) = \{0\}$.

Answers: (true/false)

(a) false; $\quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) true; $\quad \ker(A) = \{x \in \mathbb{R}^4 : Ax = 0\}$.

(c) true; $\quad A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

(d) false; $\quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(e) false; $\quad$ rank $= 1$ (reduce)

(f) true $\quad$ invertible $\Rightarrow$ $Ax = b$ has unique solution for any b

$\Rightarrow$ $Ax = 0$ has unique solution $x = 0$

$\Rightarrow$ $\ker(A) = \{0\}$.