Name: (print) Solutions

Show all your work.

1. (4 points) Find the bases of the image and the kernel of the matrix

$$A = \begin{pmatrix} 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (v_1, v_2, v_3, v_4, v_5)$$

Im(A):

$$\left.\begin{aligned} v_3 &= 5v_1 + 4v_2 \\ v_4 &= 3v_1 + 2v_2 \end{aligned}\right\} \text{ redundant}$$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ - linearly independent

$\Rightarrow$ Basis.

Ker(A):

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & 3 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ is in rref.

free variables

$$x_3 = s$$
$$x_4 = t$$
$$x_1 = -5s - 3t$$
$$x_2 = -4s - 2t$$
$$x_5 = 0$$

In vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5s - 3t \\ -4s - 2t \\ s \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} -5 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Ker}(A) = \text{span}\left\{ \underbrace{\begin{pmatrix} -5 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{linearly independent}} \right\}$$

linearly independent $\Rightarrow$ Basis.

Please turn over...

2. (2 points) Can you find a 3×3 matrix A such that $\ker(A) = \mathrm{im}(A)$? If yes give an example, if no explain why.

No. The rank-nullity theorem says that

$$\dim(\ker(A)) + \dim(\mathrm{im}(A)) = 3$$

Can't split 3 into 2 equal integers $\Rightarrow$ dimensions are not equal $\Rightarrow$ $\ker(A) \neq \mathrm{im}(A)$.

(For a 2x2 matrix the answer is yes: if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\mathrm{im}(A) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$; $\ker(A) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$)

3. (2 points) If A is a 2×2 matrix such that $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ show that A is similar to a diagonal matrix D. Find an invertible S such that $S^{-1}AS = D$.

Take $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Then $Av_1 = 3v_1$, $Av_2 = -v_2$

Take the basis $\mathcal{B} = \{v_1, v_2\}$, then

$$[Av_1]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad [Av_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$\Rightarrow$ the matrix of the transformation in this basis is

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

We must then have

$$D = S^{-1}AS, \text{ where}$$
$$S = (v_1, v_2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$