Fri, Nov. 14, 2008 **MATH 262** **Midterm Test 2**

Name: (print) Solutions

CSUN ID No. :

This test includes 6 questions in the main part (62 points), and 1 bonus question (10 points). Please check that your copy has 7 pages. The duration of the test is 50 minutes.

Your scores: (do not enter answers here)

1	2	3	4	5	6	7	total

Important: The test is closed books/notes. Graphing calculators are not permitted. Show all your steps.

1. (8 points) If the matrix of a linear transformation T in the basis $\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$ is

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}}, \left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{B}} \right)$$

find the matrix of T in the standard basis.

$\mathcal{B} = \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$; $\mathcal{A} = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ – standard basis

$\left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}a\\c\end{pmatrix}$; $\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}b\\d\end{pmatrix}$

$\left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{A}} = \begin{pmatrix}c\\a\end{pmatrix}$; $\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{A}} = \begin{pmatrix}d\\b\end{pmatrix}$

$$A = \left(\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{A}} \ \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{A}} \right) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Another way:

$$B = S^{-1}AS \Rightarrow A = SBS^{-1} =$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2. (12 points) Consider the transformation $T(f(t)) = f''(t) - f(t)$ from P_2 to P_2.

(a) Show that the transformation T is linear.

$$T(f(t)+g(t)) = (f(t)+g(t))'' - (f(t)+g(t))$$
$$= (f''(t)-f(t)) + (g''(t)-g(t))$$
$$= T(f(t)) + T(g(t)). \checkmark$$
$$T(kf(t)) = (kf(t))'' - kf(t)$$
$$= k(f''(t)-f(t)) = kT(f(t)) \checkmark$$

(b) Find the matrix of T with respect to the basis $\{1, t, t^2\}$.

$$\begin{array}{ccc} a+bt+ct^2 & \xrightarrow{T} & 2c-a-bt-ct^2 \\ \text{coord.}\downarrow & & \downarrow\text{coord} \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} & \xrightarrow{A\cdot} & \begin{pmatrix} 2c-a \\ -b \\ -c \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 2c-a \\ -b \\ -c \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(Reduce to rref: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ => columns are lin. indep.)

(c) Use part (b) to find the image, the kernel, the rank and the nullity of T.

$$\text{im}(A) = \text{span}\Big\{ \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{\text{lin. independent}} \Big\} = \mathbb{R}^3$$

$\text{im}(T) = P_2$

$\ker(A) = \{0\}$

$\ker(T) = \{0\}.$

$\text{rank}(T) = \dim(\text{im}(T)) = 3$

$\text{nullity}(T) = \dim(\ker(T)) = 0.$

(d) Is the transformation T an isomorphism? Explain.

Yes it is. $\dim(\text{domain}) = \dim(\text{co-domain})$

$\ker(T) = \{0\}$.

3. (10 points) Consider the matrix

$$A = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 4 \end{pmatrix} \end{array}$$

(a) Find the basis of $\mathrm{im}(A)$ and find the dimension of the image.

$v_1 = 0$ - redundant

$v_2 \neq c_1 v_1$ ✓

$v_3 \neq c_2 v_2$ ✓

$v_4 = v_3 - v_2$ - redundant

$v_5 = 2v_2 + 4v_4$ - redundant.

$\mathrm{im}(A) = \mathrm{span}\{v_2, v_3\} = \mathbb{R}^2$

$\dim(\mathrm{im}(A)) = 2$.

(b) Find the basis of $\ker(A)$ and find the dimension of the kernel.

$$\left(\begin{array}{ccccc|c} 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 4 & 0 \end{array}\right)$$

$$\text{rref} = \left(\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 4 & 0 \end{array}\right)$$

free   free free

$$\begin{cases} x_1 = s \\ x_2 = t + 2u \\ x_3 = -t - 4u \\ x_4 = t \\ x_5 = u \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ 2 \\ -4 \\ 0 \\ 1 \end{pmatrix}$$

Basis: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -4 \\ 0 \\ 1 \end{pmatrix}$

$\dim(\ker(A)) = 3$.

Continued...

4. (10 points) (a) Determine if the vector x is in the linear span of the vectors v_1, v_2, v_3:

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix},$$

Find c_1, c_2, c_3:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

Linear system Reduce:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 4 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{-3,\ -2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$c_1 = 1$, $c_2 = 1$, $c_3 = -1$

The system is consistent

$\Rightarrow$ x is in the span of v_1, v_2, v_3.

(b) If it is, find the coordinates of x in the basis $\{v_1, v_2, v_3\}$.

$c_1 = 1$, $c_2 = 1$, $c_3 = -1$.

5. (10 points) Consider the set Q of all 2×2 matrices A that commute with $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (i. e. satisfy $AB = BA$).

(a) Show that Q is a subspace of $\mathbb{R}^{2\times 2}$.

$$A_1 \in Q \Rightarrow \quad A_1 B = BA_1$$
$$A_2 \in Q \Rightarrow \quad + \quad A_2 B = BA_2$$
$$(A_1 + A_2) B = B(A_1 + A_2) \Rightarrow A_1 + A_2 \in Q.$$

$$A \in Q \Rightarrow \quad AB = BA; \quad \text{multiply by } k:$$
$$(kA)B = B(kA)$$
$$\Rightarrow kA \in Q$$

(b) Find a basis of Q and determine its dimension.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

$$\left.\begin{aligned} a = a + c &\Rightarrow c = 0 \\ c = c \quad &\checkmark \\ a + b = b + d &\Rightarrow a = d \\ c + d = d &\Rightarrow c = 0 \end{aligned}\right\} \quad \begin{aligned} &a, b - \text{free} \\ &c = 0 \\ &d = a \end{aligned}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Basis: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

dimension $= 2$.

Continued...

6. (12 points) State which of the following statements are true or false. (You do not need to show your work.)

   (a) If $2u + 3v + 4w = 5u + 6v + 7w$ then the vectors u, v, w are linearly dependent.

   (b) If A is a matrix with 3 rows and 4 columns then $im(A)$ is a subspace of $\mathbb{R}^4$.

   (c) Matrix I_2 is similar to $2I_2$.

   (d) If matrix A is similar to B and B is similar to C then C is similar to A.

   (e) If V is a subspace of $\mathbb{R}^{2\times 2}$ then it must contain the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   (f) The lower triangular 2×2 matrices form a subspace of $\mathbb{R}^{2\times 2}$.

   (g) There exists a 3×3 matrix A such that $im(A)$ has exactly three vectors.

   (h) If T is a linear transformation from P_6 to P_4 then the kernel must be 2-dimensional.

Answers:

(a) T ($3u + 3v + 3w = 0 \Rightarrow$ nontrivial lin. relation)

(b) F ($im(A)$ is a subspace of $\mathbb{R}^3$)

(c) F ($2I = S^{-1} I S = S^{-1}S = I$ is not possible).

(d) T ($B = S^{-1}AS$, $C = R^{-1}BR \Rightarrow C = R^{-1}S^{-1}ASR = (SR)^{-1}A(SR)$)

(e) T

(f) T

(g) F

(h) F

(any subspace must contain zero element)

$$\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & 0 \\ b+e & c+f \end{pmatrix} \text{ - lower triangular} \right.$$

$$\left. k\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ka & 0 \\ kb & kc \end{pmatrix} \text{ - lower triangular}\right)$$

($im(A)$ is a subspace, so either 1 vector (zero) or infinitely many.)

(It may be but it does not have to be

Example: $T(f(t)) = 0$

$ker(T) = P_6$

$dim(ker(T)) = 7$)

7. (bonus: 10 points) Prove the following: if the vectors v_1, v_2, v_3 in $\mathbb{R}^3$ are linearly independent then they must span $\mathbb{R}^3$.

$$A = (v_1 \; v_2 \; v_3)$$

linearly independent $\Rightarrow \ker(A) = \{0\}$.

$$\Rightarrow \operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ the system of linear equations

$$Ax = y$$

has a (unique) solution x for any $y \in \mathbb{R}^3$

$\Rightarrow$ any $y \in \mathbb{R}^3$ is in the span of $v_1, v_2 v_3$.

For another way, see Fact 3.3.4 (c) in the book.