Midterm 1 Review Questions

1. Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answers.

   (a) The set of all continuous functions on the interval $[0, 1]$;

   (b) The set of all non-negative functions on the interval $[0, 1]$;

   (c) The set of all polynomials of degree exactly n;

   (d) The set of all polynomials of degree exactly n and the zero polynomial;

   (e) The set of all polynomials of degree at most n;

   (f) The set of all symmetric $n \times n$ matrices, *i. e.* the set of matrices $A = (a_{ij})$, $i, j = 1 \dots n$ such that $a_{ij} = a_{ji}$.

2. True or false:

   (a) Every vector space contains a zero vector;

   (b) A vector space can have more than one zero vector;

   (c) An $m \times n$ matrix has m rows and n columns;

   (d) If f and g are polynomials of degree n, then $f + g$ is also a polynomial of degree n;

   (e) If f and g are polynomials of degree at most n, then $f + g$ is also a polynomial of degree at most n.

3. Suppose v and w are elements of a vector space V. Prove using axioms and showing all steps:

   (a) $(v + w) + w = v + 2w$.

   (b) $(\frac{1}{2}v + w) + \frac{1}{2}v = v + w$.

   (c) $-v$ is unique: if $v + w = v + x = 0$ then $w = x$.

   (d) $-v = (-1)v$.

   (e) $w - v = -(v - w)$.

4. Consider the set $\mathbb{R}^2$ with the operations

$$\begin{aligned}(v_1, v_2) + (w_1, w_2) &= (v_1 + w_1 - 3, v_2 + w_2 + 5)\\ r(v_2, v_2) &= (rv_1 - 3r + 3, rv_2 + 5r - 5).\end{aligned}$$

(a) Determine the additive identity ("the zero vector") and verify that the definition of the additive identity indeed holds for this vector.

(b) Given a vector $v = (v_1, v_2)$ determine its additive inverse $-v$.

(b) Verify the distributive law:

$$r(v + w) = rv + rw.$$

5. Consider the functions

$$f(x) = \cos^2 x, \quad g(x) = \sin^2 x, \quad h(x) = \cos(2x), \quad i(x) = 1.$$

(a) Express $f(x)$ as a linear combination of $g(x)$, $h(x)$ and $i(x)$.

(b) Express $g(x)$ as a linear combination of $h(x)$ and $i(x)$.

(c) Show that there are infinitely many possible linear combinations that could be used in part (a).

6. Determine whether the following subsets of $\mathbb{R}^3$ are subspaces:

(a) $S = \{(x, y, z) \in \mathbb{R}^3 : |x| = |y| = |z|\}$.

(b) $S = \{(x, y, z) \in \mathbb{R}^3 : x \leq y \leq z\}$.

(c) $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + 2xy - 2xz - 2yz = 0\}$.

7. (a) Show that the sets

$$E = \{f \in \mathscr{F}(\mathbb{R}) : f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$$

and

$$O = \{f \in \mathscr{F}(\mathbb{R}) : f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}$$

(the sets of even and odd functions on $\mathbb{R}$) are subspaces of $\mathscr{F}(\mathbb{R})$.

(b) Show that any function in $\mathscr{F}(\mathbb{R})$ can be represented as a sum of a function in O and a function in E.

8. What is the next step of applying the Gauss-Jordan algorithm to the matrices?

$$\text{(a)} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 10 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 \end{bmatrix}.$$

9. Consider the system of linear equations:

$$\begin{cases} x_1 + x_2 + 2x_3 - x_4 = 0 \\ x_1 + x_3 + 2x_4 = 0 \\ -x_1 + 2x_2 + x_3 - 8x_4 = 0 \end{cases}$$

(a) Write the augmented matrix for this linear system.

(b) Using Gauss-Jordan algorithm, find RREF.

(c) Using RREF find the general solution of the system of linear equations. Express the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + s \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + t \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

10. If possible represent the polynomials $p(x)$ given below as linear combinations of the following ones:

$$p_1(x) = x + x^2, \quad p_2(x) = 2 + x + x^3, \quad p_3(x) = x^2 + x^3.$$

If such representation is impossible give reasons why.

(a) $p(x) = 1 + x + x^2 + x^3$

(b) $p(x) = x^3 - x^2 + x + 3$

(c) $p(x) = x^3$.

Answers:

1. (a) yes; (b) no; (c) no; (d) no; (e) yes; (f) yes.

2. (a) true; (b) false; (c) true; (d) false; (e) true.

3. (c) If $v+w=v+x$ then $(-v)+(v+w)=(-v)+(v+x)$ (axioms (A1) and (A6)), then $((-v)+v)+w=((-v)+v)+x)$ (Axiom (A4)), then $(v+(-v))+w=(v+(-v))+x)$ (Axiom (A3)), then $0+w=0+x$ (Axiom (A6)), then $w+0=x+0$ (Axiom (A3)), then $w=x$ (Axiom (A5)).

4. (a) $\vec{0}=(3,5)$; (b) $-(v_1,v_2)=(6-v_1,10-v_2)$.

5. (a) $f(x)=(-1)g(x)+0h(x)+1i(x)$ is one possibility; (b) $g(x)=\left(-\frac{1}{2}\right)h(x)+\left(\frac{1}{2}\right)i(x)$; (c) Since $f=i-g$ and $2g+h-i=0$, then $f=i-g+C(2g+h-i)$, so $f(x)=(2C-1)g(x)+Ch(x)+(1-2C)i(x)$, where C is arbitrary.

6. (a) no; (b) no; (c) yes; hint: $x^2+y^2+z^2+2xy-2xz-2yz=(x+y-z)^2$

7. (b) $f(x)=\frac{1}{2}(f(x)-f(-x))+\frac{1}{2}(f(x)+f(-x))$.

8. (a) $\begin{bmatrix}1&2&0\\0&1&-1\\0&0&2\end{bmatrix}$ (b) $\begin{bmatrix}1&2&0&5\\0&2&0&2\\0&0&1&-1\end{bmatrix}$ (c) $\begin{bmatrix}1&2\\0&1\\0&0\\0&0\end{bmatrix}$ (d) $\begin{bmatrix}1&2&0&7&11\\0&1&0&0&0\\0&0&1&-1&-2\end{bmatrix}$.

9. (a) $\begin{bmatrix}1&1&2&-1&0\\1&0&1&2&0\\-1&2&1&-8&0\end{bmatrix}$; (b) $\begin{bmatrix}1&0&1&2&0\\0&1&1&-3&0\\0&0&0&0&0\end{bmatrix}$; (c) $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-2\\3\\0\\1\end{bmatrix}$.

10. $p(x)=\frac{1}{2}p_1(x)+\frac{1}{2}p_2(x)+\frac{1}{2}p_3(x)$; (b) $p(x)=-\frac{1}{2}p_1(x)+\frac{3}{2}p_2(x)-\frac{1}{2}p_3(x)$; (c) impossible: the system of linear equations with the augmented matrix

$$\begin{bmatrix}0&2&0&1\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{bmatrix}$$

has no solution since its RREF is

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$