Mon Apr 6, 2020 **MATH 262** **Quiz 2**

Name: (print) Solutions.

Each problem is worth 2 points. Show all your work.

1. (a) Show that $S = \{p \in \mathbb{P}_3 : p(-1) = 0\}$ is a subspace of $\mathbb{P}_3$.

The zero polynomial $p_0(x) = 0$ satisfies $p_0(-1) = 0 \Rightarrow p_0 \in S$

If $p_1 \in S, p_2 \in S$ then $p_1(-1) = 0, p_2(-1) = 0 \Rightarrow (p_1 + p_2)(-1) = 0$

If $r \in \mathbb{R}, p \in S$ then $p(-1) = 0 \Rightarrow r(p(-1)) = 0$
$\Rightarrow (rp)(-1) = 0$
$\Rightarrow rp \in S$

S contains the zero of $\mathbb{P}_3$, S is closed under "+" and "scalar·" $\Rightarrow$
S is a subspace of $\mathbb{P}_3$.

(b) Show that $B = \{(x+1), (x+1)^2, (x+1)^3\}$ is a basis of S.

If $p \in S$ then $p(-1) = 0 \Rightarrow p(x) = (x+1)(a_0 + a_1 x + a_2 x^2)$
($x = -1$ is a root.)

If $q(x) = a_0 + a_1 x + a_2 x^2$ then
$q(x) = b_0 + b_1(x+1) + b_2(x+1)^2$ (by Taylor polynomial, $b_0 = q(-1)$, $b_1 = q'(-1)$, $b_2 = \frac{1}{2}q''(-1)$)

So
$p(x) = b_0(x+1) + b_1(x+1)^2 + b_2(x+1)^2$.
$\Rightarrow B$ spans S.

Since
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF has one in every column
$\Rightarrow$ the set is linearly indep.
Spanning + lin. Indep = Basis.

Please turn over...

2. Let V be an n-dimensional vector space and suppose a set $S = \{v_1, \dots v_n\}$ in V is linearly independent. Prove that S must be a basis of V.

Suppose S does not span V

Then there is $v_{n+1} \in V$ such that

$v_{n+1} \neq r_1 v_1 + r_2 v_2 + \dots + r_n v_n$ (for any choice of r_i)

Then $\{v_1, \dots v_n, v_{n+1}\}$ is linearly independent

(In a combination

$c_1 v_1 + \dots + c_n v_n + c_{n+1} v_{n+1} = 0$

$c_{n+1} = 0$ implies $c_1 = c_2 \dots c_n = 0$ since $v_1 \dots v_n$ are lin. indep.

and if $c_{n+1} \neq 0$ then $v_{n+1} = -\frac{c_1}{c_{n+1}} v_1 \dots - \frac{c_n}{c_{n+1}} v_n$ which is impossible.)

But V has a basis $\{w_1, \dots w_n\}$, so by comparison theorem any lin. indep. subset of V must have $\leq n$ vectors.

Therefore S spans $V \Rightarrow S$ is a basis.

3. Suppose B is a basis of $\mathbb{R}^2$ such that

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -4 \end{bmatrix}_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find the vectors of the basis B.

$B = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\}$.

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\quad -\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\Rightarrow \begin{cases} v_1 + w_1 = 3 \\ -v_1 + w_1 = 1 \end{cases}$, $\quad \begin{cases} v_2 + w_2 = 2 \\ -v_2 + w_2 = -4 \end{cases}$

$2w_1 = 4 \Rightarrow w_1 = 2$

$\Rightarrow v_1 = 1$

$2w_2 = -2 \Rightarrow w_2 = -1$

$\Rightarrow v_2 = 3$

$B = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.