MATH 262 5/3/2020

Symmetric Matrices (8.4) 1

Def: $A \in M(n,n)$ is symmetric if
$$A^T = A$$

[A^T is the matrix with the entries a_{ji} if A has entries a_{ij}]

Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is symmetric, while

$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is not

Def: For $v, w \in \mathbb{R}^n$ define

$$v \circ w = v^T w \quad \text{(row times column)}$$

– the dot product
(inner product)

More generally, a function on $V \times V$ is called inner product if it is – bilinear
– positive definite
– symmetric

Denote $\langle v, w \rangle$ for a pair $v, w \in V$

Then we ask

$$\langle \alpha v + \beta w, z \rangle = \alpha \langle v, z \rangle + \beta \langle w, z \rangle$$

$$\langle v, w \rangle = \langle w, v \rangle$$

and $\langle v, v \rangle \geq 0$, $\langle v, v \rangle = 0$ iff $v = 0$.

<u>Example</u>: $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$$\langle v, w \rangle = v_1 w_1 + 2 v_2 w_2 + 3 v_3 w_3$$

or

$$\langle v, w \rangle = v_1 w_1 + v_1 w_2 + v_2 w_1 + 2 v_2 w_2 + 3 v_3 w_3$$

...

various functions are possible that have properties of the inner product.

We'll use the standard inner product (dot product) on $\mathbb{R}^n$:

$$\langle v, w \rangle = v \cdot w = v^T w.$$

<u>Proposition</u> (Thm 8.18)

$$Av \cdot w = v \cdot A^T w$$

<u>Proof</u>: $(Av)^T w = (v^T A^T) w = v^T (A^T w)$
$= v \cdot (A^T w)$.

Proposition (Thm 8.19)

$A \in M(n,n)$

$A = A^T \iff (Av \cdot w) = v \cdot (Aw)$

Proof: "$\Rightarrow$" follows from the previous

"$\Leftarrow$" Let $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← ith position

Then $Ae_i \cdot e_j = e_i \cdot Ae_j$

Ae_i — ith column of A

$\underbrace{Ae_i \cdot e_j}_{= a_{ji}}$ — jth entry in the ith column

Ae_j — jth column of A

$\underbrace{e_i \cdot Ae_j}_{= a_{ij}}$ — ith entry on jth column

So $a_{ij} = a_{ji} \Rightarrow A = A^T$.

<u>Def.</u> $B = \{u_1, \dots u_n\}$ – orthonormal basis if B – basis, and

$$u_i \cdot u_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

The standard basis of $\mathbb{R}^n$: $\{e_1 \dots e_n\}$ is orthonormal

If P – change of basis from B to B_0 (standard)

then $P = [w_1, \dots w_n]$

$$[P^T P]_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

So $P^T P = I$ and the inverse of P is P^T. Such a matrix is called orthogonal

<u>Def.</u> $P \in M(n,n)$ is orthogonal if $P^T P = I$ (So $PP^T = I$ as well)

The Upshot:

[Theorem 8.21] Spectral Theorem for Symmetric Matrices

If A (n,n) is symmetric then there is an orthonormal basis $B = \{w_1 .. w_n\}$ for $\mathbb{R}^n$.

If P is the matrix of change of basis from B to B_0 (the standard)

then
$$A = PDP^T \quad (\text{or } D = P^TAP)$$
where D is diagonal matrix.

Some steps in the proof
(more in the textbook)

[Theorem 8.23] The char. polynomial of a symmetric matrix has n roots, counting with multiplicity

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(A - \lambda I) = (1-\lambda)^2$

$\lambda = 1$ - root of mult. 2.

$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

$\det(A - \lambda I) = (2-\lambda)^2 - 9$
$= \lambda^2 - 4\lambda - 5$
$= (\lambda - 5)(\lambda + 1)$

- 2 distinct real roots $\lambda = -1$, $\lambda = 5$.

Proof of Thm 8.23:

If $\lambda = \alpha + \beta i$ - root, $\lambda^* = \alpha - \beta i$

λ - eigenvalue $\Rightarrow \exists\ v \neq 0$

$$(A - \lambda I) v = 0$$

$$\Rightarrow (A - \lambda^* I)(A - \lambda I) v = 0$$

$$\Rightarrow (A^2 - (\lambda^* + \lambda) A + |\lambda|^2 I) v = 0$$

$$\Rightarrow ((A^2 - 2\alpha A + \alpha^2 I) + \beta^2 I) v = 0$$

$$\Rightarrow ((A - \alpha I)^2 + \beta^2 I) v = 0$$

Then $((A - \alpha I)^2 + \beta^2 I) v \cdot v = 0$

$$\Rightarrow (A - \alpha I)^2 v \cdot v + \beta^2 |v|^2 = 0$$

$$(A - \alpha I) v \cdot (A - \alpha I) v + \beta^2 |v|^2 = 0$$

$$|(A - \alpha I) v|^2 + \beta^2 |v|^2 = 0 \Rightarrow \beta = 0.$$

since $|v|^2 \neq 0$

Next step for the Spectral Theorem

[Thm 8.26] If $A \in M(n,n)$ is symmetric

$\lambda_1 \neq \lambda_2$ - eigenvalues of A

v_1, v_2 - eigenvectors

Then v_1, v_2 - orthogonal

Proof: $\lambda_1 v_1 \cdot v_2 = Av_1 \cdot v_2 = v_1 \cdot Av_2 = \lambda_2 v_1 \cdot v_2$

$\Rightarrow (\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0 \Rightarrow (v_1 \cdot v_2) = 0$

since $\lambda_1 - \lambda_2 \neq 0$

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ - symmetric.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + 2 - 3(1-\lambda)$$

$$= \lambda^2(3-\lambda) = 0$$

$\Rightarrow \lambda = 0,\ \lambda = 3$ - eigenvalues.

$\lambda = 0$: $A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s$ - solution

$\lambda = 3$:

$$A - \lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

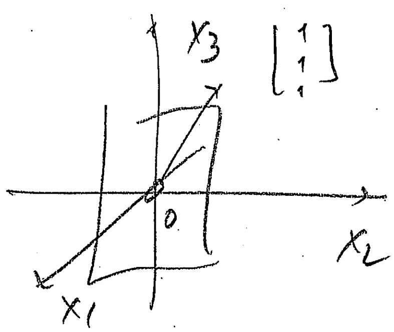
As is the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

do not form an orthonormal (or orthogonal) basis, even though the first two vectors are orthogonal to the third one.

But, let's approach this geometrically:

the first equation $(\lambda = 0)$

is $x_1 + x_2 + x_3 = 0$



The solution is a plane through the origin perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

We can pick two orthogonal vectors in that plane:

for instance $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

just find a vector that is perpendicular to both $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$:

$$\begin{bmatrix}1 & 1 & 1\\ -1 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{bmatrix} \to$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\-2\\1\end{bmatrix} t$$

Then

$$B = \left\{ \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

is orthogonal basis, and

$$B' = \left\{ \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}, \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix} \right\}$$

is orthonormal basis.

(each vector in B is normalized by $\frac{1}{\text{length}}$.)