Diagonalization (5.3)

Diagonal matrix:

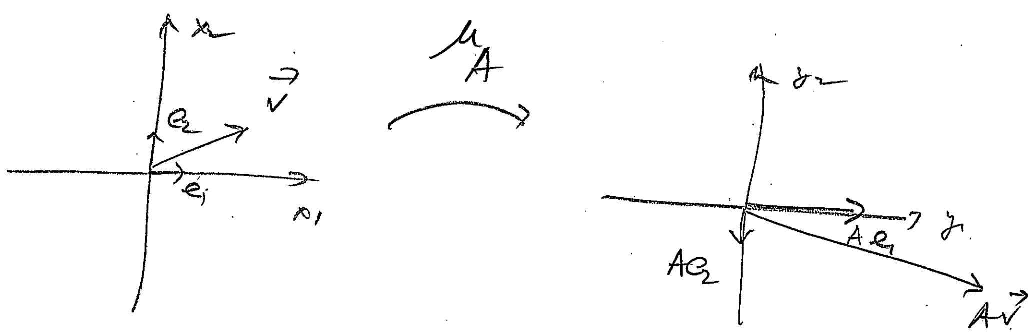
$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix};$$ $n \times n$ matrix, n nonzero entries

$\lambda_1 \cdots \lambda_n$ not necessarily all distinct, or nonzero

Ex:

$$I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$ — identity

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$



Multiplication of a vector by diagonal matrix results in scaling each component by a number λ_i
(scaling/reflection if $\lambda_i < 0$)

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \vdots \\ \lambda_n x_n \end{bmatrix}$$

Def: A matrix $A \in M(n,n)$ is diagonalizable, if there is a diagonal matrix $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ and an invertible matrix P such that $D = P^{-1}AP$.

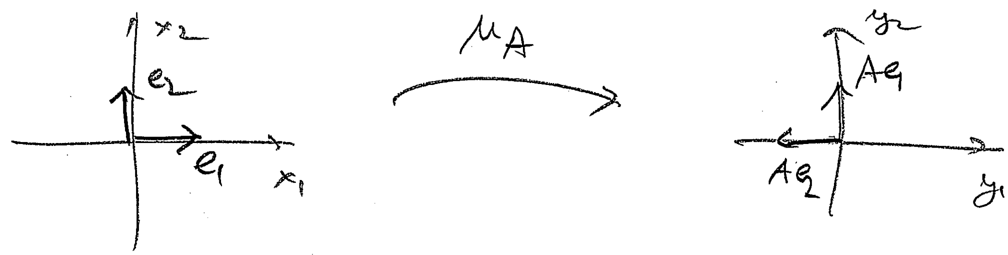
In other words, a square matrix is diagonalizable if it is <u>similar</u> to a diagonal matrix

Not every matrix is diagonalizable.

Ex:

(i) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$

rotation through 90°;



There is no vector that is transformed into scalar multiple of itself $\Rightarrow$ no (real) eigenvectors

Eigenvalues:

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad \text{— no real roots}$$

(Eigenvalues are complex $\pm i$)

(ii) $$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$

$\lambda = 1$ is an eigenvalue of multiplicity 2

No other eigenvalues (real or complex)

Eigenvectors: $A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

No other eigenvectors; can't find a basis of eigenvectors.

[Theorem 8.12–8.13] An $n \times n$ matrix A is diagonalizable $\iff$ there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of the matrix A.

Proof: "$\Rightarrow$" If $P^{-1}AP = D$

then $AP = PD$

The matrix product AP consists of columns $[Aw_1 ; Aw_2 ; \ldots Aw_n]$

where $w_1, w_2 \ldots w_n$ are columns of P.

The product PD is $[\lambda_1 w_1 ; \lambda_2 w_2 ; \dots ; \lambda_n w_n]$.

So $Aw_1 = \lambda_1 w_1$, $Aw_2 = \lambda w_2$ ... $Aw_n = \lambda w_n$

Columns of P are eigenvectors;
Columns of P are linearly independent since P is invertible

$\Rightarrow$ $\mathbb{R}^n$ has a basis of eigenvectors (columns of P.)

"$\Leftarrow$" Conversely, if $w_1, \dots w_n$ - basis of eigenvectors, take

$$P = [w_1 \dots w_n]$$

then P is invertible since the columns of P are lin. independent

In the basis $w_1 \dots w_n$ the matrix of μ_A is

$$A' = P^{-1} A P = D$$

since $Aw_i = \lambda_i w_i$.

[Theorem 8.14] Eigenvectors corresponding to distinct eigenvalues of A are linearly independent.

Moreover, if $\lambda_1 \dots \lambda_m$ - eigenvalues

$$\underbrace{v_{11} \dots v_{1k_1}}_{\text{lin. indep.}} \quad \dots \quad \underbrace{v_{m1} \dots v_{mk_m}}_{\text{lin. indep}}$$

Then the set of vectors listed above is linearly independent.

Idea of proof: $(m=3)$

Suppose $r_1 v_1 + r_2 v_2 + r_3 v_3 = 0$

Apply A: $r_1 A v_1 + r_2 A v_2 + r_3 A v_3 = 0$

$$r_1 \lambda_1 v_1 + r_2 \lambda_2 v_2 + r_3 \lambda_3 v_3 = 0$$

multiply by λ_3 the first equation: $-$ $r_1 \lambda_3 v_1 + r_2 \lambda_3 v_2 + r_3 \lambda_3 v_3 = 0$

$\Rightarrow$ $c_1 v_1 + c_2 v_2 = 0$

where $c_1 = r_1(\lambda_3 - \lambda_1)$

$c_2 = r_2(\lambda_3 - \lambda_1)$

Apply A: $c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$

$\Rightarrow$ $c_1 \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} v_1 = 0 \Rightarrow c_1 = 0$

$c_2 \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} v_2 = 0 \Rightarrow c_2 = 0$

Since $\lambda_3 - \lambda_1 \neq 0$ also $r_1 = r_2 = 0$

$$\Rightarrow \quad r_3 \underbrace{v_3}_{\neq 0 \text{ (eigenvector)}} = 0 \qquad \Rightarrow r_3 = 0 .$$

Example:
(p. 318)

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

eigenvalues
4, 7, 2 - distinct
one eigenvector per eigenvalue
gives a basis for $\mathbb{R}^n$
$\Rightarrow$ diagonalizable.

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

eigenvalues 4, 7

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ e/v for $\lambda = 4$

$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ e/v for $\lambda = 7$

no basis of eigenvectors
$\Rightarrow$ not diag.

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

eigenvalues 4, 7

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ e/v for $\lambda = 4$

$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ e/v for $\lambda = 7$

3 lin. indep. eigenvectors $\Rightarrow$ diagonalizable.