MATH 262 4/29/2020

Similarity (8.2)

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \\
V \arrow[u, leftrightarrow, "{[\ ]_B}"] \arrow[d, leftrightarrow, "{[\ ]_{B'}}"'] \arrow[r, "T"] & V \arrow[u, leftrightarrow, "{[\ ]_B}"'] \arrow[d, leftrightarrow, "{[\ ]_{B'}}"] \\
\mathbb{R}^n \arrow[r, "A'"] & \mathbb{R}^n
\end{tikzcd}$$

$B = \{v_1 \dots v_n\}$

$B' = \{w_1 \dots w_n\}$

P – $n \times n$ matrix that transforms coordinates on basis B' to coordinates on basis B

$$[v]_B = P[v]_{B'}$$

i.e. $P = \left([w_1]_B, \dots [w_n]_B \right]$

Then $A' = P^{-1} A P$

(P^{-1} is $n \times n$ matrix that transforms coordinates on basis B to coordinates on basis B'.)

P^{-1} is the inverse of P.

Now suppose the basis B' consists of eigenvectors of T: $T(w_i) = \lambda_i w_i$.

Then
$$A' = \left[[T(w_1)]_{B'}, \ldots, [T(w_n)]_{B'} \right]$$
$$= \left[[\lambda_1 w_1]_{B'}, \ldots [\lambda w_n]_{B'} \right]$$
$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \mathrm{diag}[\lambda_1 \ldots \lambda_n]$$

diagonal matrix with $\lambda_1 \ldots \lambda_n$ on the diagonal

<u>Def</u>: Suppose $A, A' \in M(n,n)$. A' is <u>similar</u> to A if there is an invertible matrix P such that $P^{-1}AP = A'$.

<u>Remark</u>: $P^{-1}AP = A' \Rightarrow P(P^{-1}AP) = PA'$
$$\Rightarrow AP = PA' \Rightarrow (AP)P^{-1} = (PA')P^{-1}$$
$$\Rightarrow A = PA'P^{-1}$$

If $Q = P^{-1}$ then $A = Q^{-1}A'P^{-1}$

So A' similar to $A \Rightarrow A$ is similar to A'.

Further if A similar to B, B similar to C

then $A = P^{-1}BP$, $B = M^{-1}CM \Rightarrow$
$$A = P^{-1}(M^{-1}CM)P = (P^{-1}M^{-1})C(MP) = (MP)^{-1}C(MP)$$

so necessarily A is similar to C.

Finally, $A = I^{-1} A I$ so A is similar to itself.

Similarity of matrices is an equivalence relation.

[Thm 8.5, Exercise 1].

[Theorem 8.8] Similar matrices have the same characteristic polynomial

If $A' \sim A$ (similar) then $\det(A-\lambda I) = \det(A'-\lambda I)$

Proof:

$$A' = P^{-1}AP$$
$$I = P^{-1}P$$
$$\lambda I = P^{-1} \cdot (\lambda I) \cdot P$$
$$\begin{aligned}(A'-\lambda I) &= P^{-1}AP - P^{-1}(\lambda I)P \\ &= P^{-1}(AP - \lambda I \cdot P) \\ &= P^{-1}(A-\lambda I)P\end{aligned}$$

So $\det(A'-\lambda I) = \det(P^{-1})\det(A-\lambda I)\det(P)$

$$= \det(A-\lambda I).$$

Notice:

$$p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

– char. polynomial

From this form, see that $p(\lambda)$ is a polynomial of degree n, with leading term $(-1)^n \lambda^n$.

Also when $\lambda = 0$, $p(\lambda) = \det(A)$.

$\Rightarrow \det(A)$ is the free term (the constant term) in the polynomial $p(\lambda)$

Furthermore, the term with the power λ^{n-1} is $(-1)^{n-1} tr(A) \lambda^{n-1}$

where

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

– Sum of the diagonal entries of the matrix, known as the <u>trace</u> of A.

<u>Remark</u> [Theorem 8.8] $\Rightarrow$ Char. polynomial is well-defined for any operator $T: V \to V$ on a finite-dimensional space:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow, "{[\,]_B}"'] & V \arrow[d, leftrightarrow, "{[\,]_B}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

$p(\lambda) = \det(A - \lambda I)$ is independent of the choice of the basis.

<u>Def:</u> A lin. operator $T: V \to V$ (V is finite-dimensional) is diagonalizable

$\Leftrightarrow$ its matrix is similar to a diagonal matrix

If $B' = \{w_1 \dots w_n\}$ is a basis in which the matrix of T is diagonal,

$$e_i = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \text{ pos} \\ \\ \\ \end{array} \end{array}$$

Then

$$[T(w_i)]_{B'} = \lambda_i e_i = \lambda_i [w_i]_{B'}$$

So the vectors of the basis B' must be eigenvectors of T.

<u>Example</u>: $V = \mathbb{P}_2 = \{ a_0 + a_1 x + a_2 x^2 : a_i \in \mathbb{R} \}$

$T : V \longrightarrow V$ is the differentiation operator

$$T(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x .$$

Since $T(a_0) = 0 = 0 \cdot (a_0)$

$\lambda_1 = 0$ is an eigenvalue, and

$p_1(\lambda) = 1$ is the corresponding eigenvector.

In the basis $B = \{1, x, x^2\}$,

T is represented by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3$$

so $\lambda = 0$ is the only eigenvalue

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ has rank 2

the eigenspace is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

There is no basis of eigenvectors $\Rightarrow$

T is <u>not</u> diagonalizable.

Example: $A=\begin{bmatrix}1&2\\0&3\end{bmatrix}$; $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T=\mu A$.

$$|A-\lambda I| = \begin{vmatrix}1-\lambda & 2\\ 0 & 3-\lambda\end{vmatrix} = (1-\lambda)(3-\lambda)$$

roots are $\lambda=1$, $\lambda=3$ - eigenvalues.

$\underline{\lambda=1}$: $A-\lambda I = \begin{bmatrix}0&2\\0&2\end{bmatrix} \Rightarrow w_1=\begin{bmatrix}1\\0\end{bmatrix}$

is an eigenvector

$\underline{\lambda=3}$: $A-\lambda I = \begin{bmatrix}-2&2\\0&0\end{bmatrix} \Rightarrow w_2=\begin{bmatrix}1\\1\end{bmatrix}$

is an eigenvector

vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ are linearly independent $\Rightarrow$ they form a basis of $\mathbb{R}^2$. ($\mathbb{R}^2$ is 2-dimensional!!)

$B_0=\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

In the basis $B'=\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ the matrix of T in this basis is

$$A' = \begin{bmatrix}1&0\\0&3\end{bmatrix}$$

So $A=\begin{bmatrix}1&2\\0&3\end{bmatrix}$ is diagonalizable, and

$P^{-1}AP = A'$, where $P=\begin{bmatrix}1&1\\0&1\end{bmatrix}$.