Properties of Determinants (Continued)

(7.3 and beginning of 7.

1. Products [Thm 7.7]

$$\det(AB) = \det(A)\det(B)$$

[Note: This kind of rule does not work for sums/differences:

$$\det\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\right) = \det\left(\begin{bmatrix}0&0\\0&0\end{bmatrix}\right) = 0$$

but $\det\begin{bmatrix}1&0\\0&1\end{bmatrix}+\det\begin{bmatrix}-1&0\\0&-1\end{bmatrix} = 1+1 = 2$]

Idea of Proof: A can be brought to diagonal form by a sequence of elementary row operations:

1. Interchange two rows
2. Add a multiple of one row to another row.

These operations can be realized as matrix multiplicatio

$$\underset{\text{row interchange}}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}c&d\\a&b\end{bmatrix}; \quad \underset{\substack{\text{add } k\text{-multiple of}\\ \text{row 1 to row 2}}}{\begin{bmatrix}1&0\\k&1\end{bmatrix}}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}a&b\\ka+c&kb+d\end{bmatrix}$$

Same works for matrices of larger size:

$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ interchanges row 1 with row 3...

Apply elementary row operations to A:

$$E_m E_{m-1} \cdots E_2 E_1 A = D \quad \text{- diagonal}$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}, \text{ so } DB = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} d_1 b_{11} & \cdots & d_1 b_{1n} \\ \vdots & & \\ d_n b_{n1} & \cdots & d_n b_{nn} \end{pmatrix}$$

Then

$$\det(AB) = \det(E_m \cdots E_1 AB)$$

(# of row interchanges) $\longrightarrow$

$$= (-1)^k \det(DB) = (-1)^k \underline{d_1 \cdots d_n} \det(B)$$

and

$$\det(A) = \det(E_m \cdots E_1 A) =$$

$$= (-1)^k \det(D) = (-1)^k \underline{d_1 \cdots d_n}$$

So $\det(AB) = \det(A)\det(B)$.

2. <u>Inverses</u>

(i) If A is invertible,

$$AA^{-1} = I, \quad A^{-1}A = I$$

$$\det(AA^{-1}) = \det(I) = 1$$

$$\det(A)\det(A^{-1}) = 1$$

so $\det(A) \neq 0$, $\quad \det(A^{-1}) = \dfrac{1}{\det(A)}$.

Also if $\det(A) = 0$ then $AA^{-1} = I$ or $A^{-1}A = I$ are impossible,

so A cannot be invertible.

(ii) If $\det(A) \neq 0$ then A is invertible.

Proof: By contradiction: Suppose A is not invertible.

Then RREF cannot be $\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$

So RREF must have a zero row.

The determinant for RREF is then zero, and since RREF is obtained from A by a sequence of row operations

$$\det(A) = \det(E_m \cdots E_1 D) = C \det(D)$$
$$= 0$$

Thus, A is invertible $\iff \det(A) \neq 0$

A is not invertible (singular) $\iff \det(A) = 0$.

Eigenvalues and Eigenvectors (8.1)

Ex: $T: \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 23 & -12 \\ 40 & -21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

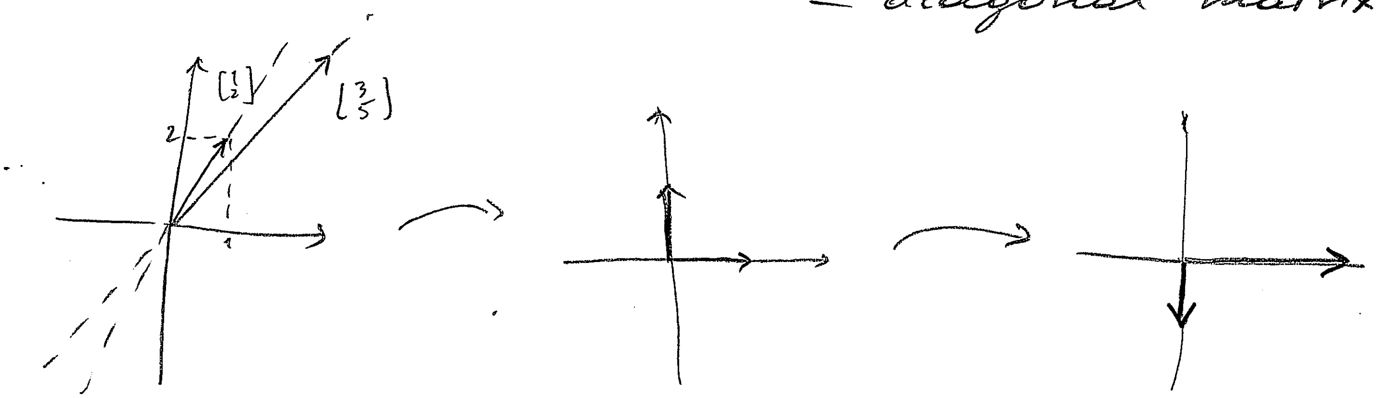
$$T\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

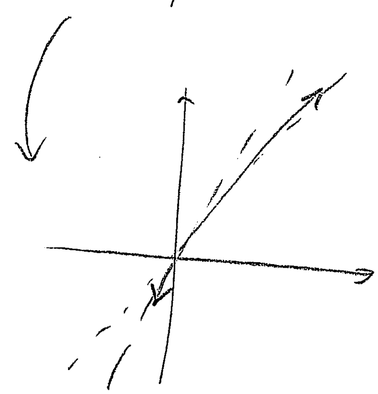
So, relative to basis $B = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ the matrix of T is

$$\left[\left[T\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right]_B ; \left[T\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right]_B \right] = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

— diagonal matrix.



Seen in the basis B, the action of T is particularly simple: first coordinate direction is stretched by a factor of 3, the second coordinate direction reflected through the origin.



Goal: to find a basis B for a matrix A of a lin. transf. T such that on this basis

$$A' = P^{-1}AP \text{ is diagonal}$$

(P changes from B - standard to B_0 - standard basis)

Def: $T: V \to V$ lin. operator

$v \in V$ is called an eigenvector $v \neq 0$ if $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$.

$$E_\lambda = \{v \in V: T(v) = \lambda v\} \text{ is}$$

an eigenspace

Characteristic equation

[Thm 8.2] $\lambda \in \mathbb{R}$ is an eigenvalue iff $\det(A - \lambda I) = 0$.

Proof: If $T(v) = \lambda v$

$$[T(v)]_B = A[v]_B = \lambda [v]_B$$

$$(A - \lambda I)[v]_B = 0$$

$$\det(A - \lambda I) = 0$$

Example
(p. 306)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

Find eigenvalues eigenvectors.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-\lambda)(5-\lambda) + 4 + 8 - \left(4\lambda + 2(5-\lambda) - 4(1-\lambda)\right)$$

$$= -\lambda(\lambda-5)(\lambda-1) + 12 - (6\lambda + 10 - 4)$$

$$= -\lambda(\lambda-5)(\lambda-1) - 6\lambda + 6$$

$$= (\lambda-1)\left(-\lambda(\lambda-5) - 6\right) = (\lambda-1)\left(-\lambda^2 + 5\lambda - 6\right)$$

$$= -(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

roots are $\lambda = 1$, $\lambda = 2$, $\lambda = 3$.

Eigenvectors: Solve $(A-\lambda I)x = 0$

$\lambda = 1$:

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & -4 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} t$$

Any vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} t$, $t \neq 0$ is an eigenvector

$\underline{\lambda = 2}$:

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 4 & -4 & 3 & 0 \end{array}\right]$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 4 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} t$$

Any vector $\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} t$, $t \neq 0$ is an eigenvector for $\lambda = 2$.

$\underline{\lambda = 3}$:

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right]$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} t$$

Any vector $\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} t$, $t \neq 0$ is an eigenvector for $\lambda = 3$

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\}$$ – set of three eigenvectors forms a basis of $\mathbb{R}^3$!