Properties of Determinants (7.3)

Def: $A \in M(n,n)$ $\left(\begin{matrix} n\times n \\ \text{matrix}\end{matrix}\right)$ Expansion on row 1

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

where A_{1j} is obtained by deleting row 1, column j

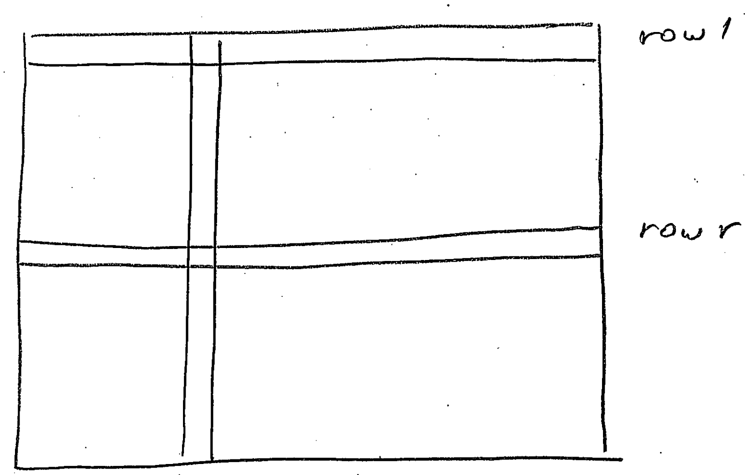
[Theorem 7.5] Expansion can be done on any row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{r+j} a_{rj} \det(A_{rj})$$

for any $r = 1, 2 \ldots n$.

(This is important because other properties of determinants can be derived from it.)

Proof



$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}) =$$

Assuming it works for smaller matrix A_{1j}

in A_{1j} row $r-1$ has elements a_{rk}

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \left(\sum_{k=1}^{j-1} (-1)^{(r-1)+k} a_{rk} \det\left(A_{1j,rk}\right) \right.$$

column j is skipped in matrix A_{1j}

$$\left. + \sum_{k=j+1}^{n} (-1)^{(r-1)+(k-1)} a_{rk} \det\left(A_{1j,rk}\right) \right)$$

Compute the right-hand side of the formula and compare:

(does not matter what name is used for summation index.)

$$\sum_{k=1}^{n} (-1)^{r+k} a_{rk} \det\left(A_{rk}\right) =$$

$$= \sum_{k=1}^{n} (-1)^{r+k} a_{rk} \left(\sum_{j=1}^{k-1} (-1)^{1+j} a_{1j} \det\left(A_{rk,1j}\right) \right.$$

$$\left. + \sum_{j=k+1}^{n} (-1)^{1+(j-1)} a_{1j} \det\left(A_{rk,1j}\right) \right)$$

Both expressions can be written as

$$\sum_{k=1}^{n} \sum_{j=1}^{n} (-1)^{r+j+k} \operatorname{sign}(k<j)\, a_{1j}\, a_{rk} \det\left(A_{rk,1j}\right)$$

where

$$\operatorname{sign}(k<j) = \begin{cases} 1, & k<j \\ 0, & k=j \\ -1, & k>j \end{cases}$$

<u>Corollaries</u> (1) If a matrix has a row of zeros, the determinant is zero.

<u>Proof</u>: Use expansion on the zero row.

(2) If two rows in a matrix are interchanged, the determinant switches sign.

(i) Suppose it's row 1 and row 2.
Let $A^{(12)}$ denote matrix with 2 interchanged rows.
Calculate $\det(A^{(12)})$ expanding on second row:

$$\det(A^{(12)}) = \sum_{j=1}^{n} (-1)^{2+j} a_{1j} \det(A_{1j})$$
$$= -\sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$
$$= -\det(A).$$

(ii) Same calculation works of we use any two adjacent rows. (i and $i+1$)

(iii) To interchange row i with row $i+m$ we do $(m-1)$ interchanges to place the two rows together, then 1 interchange to switch their order, then $(m-1)$ interchange to place $(i+m)$-th row into ith position

After $2(m-1)+1 = 2m-1$ interchanges

the sign of the determinant switches to the opposite.

(3) If a matrix has two identical rows, the determinant is zero.

Proof: Let $\det(A) = D$.

Interchange the two identical rows, for the matrix A' with the interchange

$$\det(A') = -D$$

Since $A' = A$, $D = -D \Rightarrow D = 0$.

(4) If a row in a matrix is multiplied by a constant, the constant is pulled out

Proof: Expand on the row (r) multiplied by c:

$$\det(A') = \sum_{j=1}^{n} c a_{rj} (-1)^{r+j} \det(A_{rj})$$

$$= c \sum_{j=1}^{n} a_{rj} (-1)^{r+j} \det(A_{rj})$$

$$= c \det(A)$$

(5) If a multiple of one row is added to another row, the determinant does not change.

Use row 1 as example:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \longrightarrow \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{r1}+ca_{11} & a_{r2}+ca_{r1} & \cdots & a_{rn}+ca_{rn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Expand on row r:

$$\sum_{k=1}^{n} (a_{rk}+ca_{1k})(-1)^{r+k} \det(A_{rk}) =$$

$$= \sum_{k=1}^{n} a_{rk}(-1)^{r+k} \det(A_{rk}) + c\sum_{k=1}^{n} a_{1k}(-1)^{r+k} \det(A_{rk})$$

$$= \det(A) + c \underbrace{\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}}_{\text{matrix with 2 identical rows. determinant } = 0}$$

(b) Can be proved with the same technique as Thm 7.5:

[Thm 7.6] The determinant can be computed by expansion along a column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det(A_{ik}).$$

for any $k = 1, \dots n$.

<u>Example:</u>

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 3 \end{bmatrix}; \text{ Compute } \det(A).$$

(1) By definition:

$$\det(A) = 1 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} - 5 \begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= (9-4) - 5(6-12) + 2(2-9)$$

$$= 5 + 30 - 14 = 21$$

(2) By using the 3X3 rule:

$$\left|\begin{matrix} 1 & 5 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 3 \end{matrix}\right| \begin{matrix} 1 & 5 \\ 2 & 3 \\ 3 & 1 \end{matrix} = 9 + 60 + 4 - 18 - 4 - 30$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

$$= 73 - 52 = 21$$

(3) By using Gaussian elimination:

$$\begin{vmatrix} 1 & 5 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 \\ 0 & -7 & 0 \\ 0 & -14 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 \\ 0 & -7 & 0 \\ 0 & 0 & -3 \end{vmatrix} =$$

$-2R_1 + R_2 \to R_2$

$-3R_1 + R_3 \to R_3$

$-2R_2 + R_3 \to R_3$

upper triangular matrix.

$$= 1 \cdot (-7) \cdot (-3) = 21.$$