Determinants: Definition (7.2)

$$\begin{cases} ax+by=r \\ cx+dy=s \end{cases} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

has unique solution $\iff ad-bc \neq 0$

In that case

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

The quantity $ad-bc$ is called a determinant of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We would like to generalize this concept to the case of $\mathbb{R}^n$ with an arbitrary $n \geq 1$.

For $n=1$

$$ax=r$$

has unique solution $\iff a \neq 0$

Def. Suppose $A \in M(n,n)$; $A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

$\underline{n=1}$: $\det A = a_{11}$

$\underline{n=2}$ $\det A = a_{11}a_{22} - a_{12}a_{21}$

<u>general n:</u>

<u>Def</u> $\longrightarrow$ $$\boxed{\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})}$$

where A_{1j} is the submatrix of A ~~obtained~~ by deleting the first row and the j-th column.

<u>Note</u>: This is a <u>recursive</u> definition: the size n determinant is reduced to n size $(n-1)$ determinants.

Since determinant of size 1 includes 1 term, the determinant of size n will include

$$S(n) = n \cdot S(n-1) = n(n-1)\, S(n-2)$$
$$= \dots = n(n-1) \dots 2 \cdot S(1)$$
$$= n(n-1) \dots 2 \cdot 1 = n!$$

$\rightarrow$ $n!$ terms! (very large when n is large)

$$\begin{bmatrix} a_{11} & \dots & a_{1j-1} & a_{1j} & a_{1j+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j-1} & a_{2j} & a_{2j+1} & \dots & a_{2n} \\ \vdots & & & & & & \vdots \\ a_{n1} & \dots & a_{nj-1} & a_{nj} & a_{nj+1} & \dots & a_{nn} \end{bmatrix}$$

the matrix A_{1j} is what remains of A after deleting 1 row and 1 column.

Example:

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\det(A) = 1 \cdot \det\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 2 \det\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + 3 \det\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 1 \cdot (-1) - 2(-1) + 3(-1) = -2 \neq 0$$

Reduce:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix is <u>invertible</u>

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\det(A) = 1 \cdot \det\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 2 \det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 3 \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -1 - 2 + 3 = 0$$

Reduce:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix A is <u>not</u> invertible.

<u>Notation</u>: $\det \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \vdots & \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix} = \begin{vmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{n1}\ a_{n2} \cdots a_{nn} \end{vmatrix}$

(bars do not mean "absolute value")

<u>Technique</u>: The definition of the determinant is known as

"expansion on the first row"

Similarly, we can expand on any row of the matrix, if we follow the pattern:

$$\begin{array}{c|ccccc|} & + & - & + & \cdots & (-1)^{n+1} \\ + & a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ - & a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ + & a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & & & \\ (-1)^{n+1} & a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array}$$

So $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij})$

(for any i)

— formula of expansion on i-th row.

The determinant of transposed matrix is the same as the original matrix:

$$\det(A^T) = \det(A).$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ a_{13} & a_{23} & \cdots & a_{n3} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

(rows are interchanged with columns.)

Example:

$$A = \begin{bmatrix} -2 & 8 & 7 & 4 \\ 0 & 3 & 15 & 17 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\begin{vmatrix} -2 & 8 & 7 & 4 \\ 0 & 3 & 15 & 17 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 5 \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 & 0 \\ 8 & 3 & 0 & 0 \\ 7 & 15 & 1 & 0 \\ 14 & 17 & 9 & 5 \end{vmatrix}$$

$$= (-2)\begin{vmatrix} 3 & 0 & 0 \\ 15 & 1 & 0 \\ 17 & 9 & 5 \end{vmatrix} = (-2)3\begin{vmatrix} 1 & 0 \\ 9 & 5 \end{vmatrix} =$$

$$= (-2)\cdot 3\cdot 1\cdot 5 = -30$$

Example:

$$\begin{vmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 1 & -1 \end{vmatrix} = -\left(-\begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 2 & -1 \end{vmatrix} \right)$$

(Expand along the second row)

$$= -\left(\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 4 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \right)$$

$$+ \left(\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \right)$$

$$= -\left(-7 + 18 - 10 \right)$$

$$+ \left(-9 - 0 \right)$$

$$= -1 - 9 = -10.$$

Note: In practice a direct computation for a matrix with general (nonzero) elements becomes tedious for a matrix 4×4 and more, and manual calculation is not practical.

Relation with Gaussian Elimination

[Theorem 7.4]

(1) If A has a row of zeros, $\det(A)=0$

(2) If A is identity matrix of size n,

$$A=\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \quad \text{then } \det(A)=1$$

(3) If two rows are interchanged determinant switches sign

(4) If a multiple of one row is added to another row, the determinant is unchanged

(5) If <u>all</u> elements on one row are multiplied by the <u>same</u> constant, the determinant is multiplied by the same constant.

<u>Example:</u> Compute using Gaussian Elimination!

$$\begin{vmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 4 & 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -5 & -9 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -5 & -9 \end{vmatrix}$$

$-2R_1 + R_4 \to R_4$ $\qquad$ $-2R_3 + R_1 \to R_1$

$$= \begin{vmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -5 & -9 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -5 & -9 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{vmatrix}$$

$-R_2 + R_1 \to R_1$ $\qquad$ $\frac{1}{2} R_1 \to R_1$ $\qquad$ $9R_1 + R_4 \to R_4$

$$= 2(-5) \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -10 \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$-\frac{1}{5} R_4 \to R_4$

$-R_1 + R_3 \to R_3$

$-R_4 + R_3 \to R_3$

$-R_4 + R_2 \to R_2$

$$= -10 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-10) \cdot 1 = -10,$$

$R_1 \leftrightarrow R_3$

$R_3 \leftrightarrow R_4$

The determinant of the identity matrix is 1

Gauss' method becomes more practical for calculation of determinants of 4X4 and more (unless a matrix has a lot of zeros)

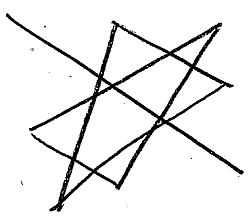
Simplified rules for 2×2 and 3×3:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{12} - a_{21}a_{12}$$

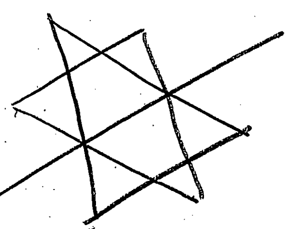
$-$ $+$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{23} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}$$

The "+" pattern:



The "−" pattern:



Alternatively:

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ - & - & - & & \\ & & + & + & + \end{array}$$

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot5\cdot9 + 3\cdot4\cdot8 + 2\cdot6\cdot7 - 3\cdot5\cdot7 - 2\cdot4\cdot9 - 1\cdot6\cdot8$$

$$= 45 + 96 + 84 - 105 - 72 - 48$$

$$= (45-48) + (96-105) + (84-72)$$

$$= -3 - 9 + 12 = 0$$

Alternatively, use "Gauss elimination!"

$$\begin{array}{l} -R_1+R_2\to R_2 \\ -R_1+R_3\to R_3 \end{array} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

$$-2R_2 + R_3 \to R_3$$