Rank and Nullity (6.7).

$T: V \to W$ - linear

Def: $\dim(\ker(T)) = \text{nullity}(T)$
(If the kernel of T is finite-dimensional)

$\dim(\text{Im}(T)) = \text{rank}(T)$
(If the image is finite-dim.)

Note: For a matrix $A \in M(m,n)$, defined rank (A) as the number of leading ones in RREF.

But this is precisely the dimension of the image, according to the method to find the basis of $\text{Im}(T)$!

The more the nullity, the more "collapsing to zero" is happening, so the less the rank, and vice versa. More precisely:

Theorem: (Dimension Theorem)
$T: V \to W$ - linear, $\dim(V)$ is finite.
Then $\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

<u>Proof</u>: $\ker(T)$ – subspace of V

V is finite-dimensional:
has a basis $\{v_1 \dots v_n\}$

If $\ker(T) = 0$ then $B_0 = \emptyset$ is a basis

Otherwise we pick a nonzero vector in $\ker(T)$ and start "expanding" to a basis of $\ker(T)$ by adding lin. indep. vectors.

The "expansion" process must stop before we reach n vectors (Comparison Theorem)

Let's say we end up with

$\{w_1 \dots w_m\}$ – basis for $\ker(T)$.

This set can be further "expanded" into a basis of V:

$$\{w_1 \dots w_m, w_{m+1} \dots w_n\}$$

We claim that $\{T(w_{m+1}) \dots T(w_n)\}$ is a basis of $\operatorname{Im}(T)$.

Indeed, these vectors are lin. indep:

$$r_{m+1} T(w_{m+1}) + \dots + r_n T(w_n) = 0$$

$$\Rightarrow T(r_{m+1} w_{m+1} + \dots + r_n w_n) = 0$$

$\Rightarrow r_{m+1} w_{m+1} + \dots + r_n w_n$ is in $\ker(T)$.

$r_{m+1}w_{m+1} + \cdots + r_n w_n = c_1 w_1 + \cdots + c_m w_m$

$\Rightarrow$ $w_1 \cdots w_n$ is lin-dependent, which is impossible.

So, the vectors $T(w_{m+1}) \ldots T(w_n)$ are lin-independent, and also

$\forall\, z \in Im(T)$ $\quad z = T(v)$

$\Rightarrow \quad z = T(a_1 w_1 + \cdots + a_n w_n)$

$= \underbrace{T(a_1 w_1 + \cdots + a_m w_m)}_{=0} + T(a_{m+1} w_{m+1} + \cdots + a_n w_n)$

since $w_1 \ldots w_m \in \ker(T)$

$= a_{m+1} T(w_{m+1}) + \cdots + a_n T(w_n)$.

So, any vector on the image of T is a linear combination of

$T(w_{m+1}), \ldots T(w_n)$

$\Rightarrow$ these vectors span $im(T)$

$\Rightarrow$ they form a basis for $im(T)$.

Since a basis of $ker(T)$ has m vectors and a basis of $im(T)$ has $n-m$ vectors,

$\dim(\ker(T)) + \dim(im(T)) = m + (n-m) = n$
$= \dim(V)$. $\square$

Theorem: A linear transformation $T: V \to W$ is one-to-one

$\iff \ker(T) = \{0\}$.

Proof: "$\Leftarrow$" Suppose $\ker(T) = 0$.

Then $T(v) = 0 \iff v = 0$.

then $T(v_1) = T(v_2) \Rightarrow T(v_1) - T(v_2) = 0$

$\Rightarrow T(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$.

(one-to-one)

"$\Rightarrow$" Suppose T is one-to-one.

Since $T(0) = 0$ then $T(v) = 0$ implies $v = 0$.

so $\ker(T) = \{0\}$.

Example: #5: Suppose $T: V \to W$ is linear and onto.

Suppose $\{v_1 \ldots v_n\}$ span V

Show that $\{T(v_1), \ldots T(v_n)\}$ span W.

Solution: Onto: $Im(T) = W$

$w \in W \Rightarrow w = T(v)$ for some $v \in V$

$\Rightarrow w = T(c_1 v_1 + \ldots + c_n v_n)$

$= c_1 T(v_1) + \ldots + c_n T(v_n)$

So $\{T(v_1) \ldots T(v_n)\}$ span W.

#6: Suppose $T: V \to W$ is linear and $v_1, \ldots v_n$ s.t. $T(v_1), \ldots T(v_n)$ are lin. indep.

Then $v_1 \ldots v_n$ are lin. indep.

Proof: $c_1 T(v_1) + \ldots + c_n T(v_n)$ possible only when $c_1 \ldots c_n = 0$.

Suppose $r_1 v_1 + \ldots + r_n v_n = 0$, not all $r_i = 0$.

Then $T(r_1 v_1 + \ldots + r_n v_n) = 0$

$\Rightarrow r_1 T(v_1) + \ldots + r_n T(v_n) = 0$

but this is only possible when $r_1, \ldots r_n = 0$.

Exercises 6.7: 3, 4, 8