Image and Kernel (6.6)

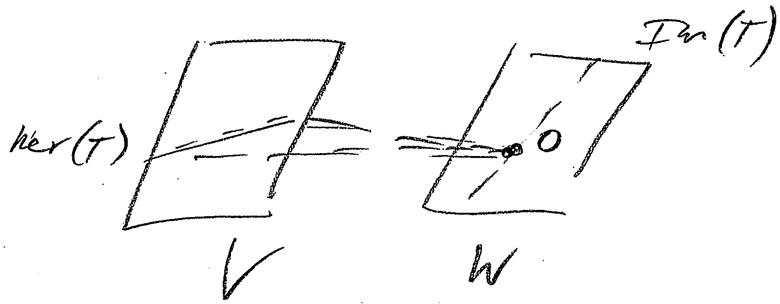
$T : V \to W$ - linear

Def: kernel (T) or nullspace (T) is the set

$$\ker(T) = N(T) = \{ v \in V : T(v) = 0 \}$$

Def: image (T) is the set

$$\operatorname{im}(T) = R(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \}$$



Proposition (Theorem 6.20)

$\ker(T)$ is a subspace of V

$\operatorname{im}(T)$ is a subspace of W.

Proof: if $v_1, v_2 \in V$ then $T(v_1) = T(v_2) = 0$

$\Rightarrow T(v_1 + v_2) = T(v_1) + T(v_2) = 0$

also $T(r v_1) = r T(v_1) = 0$.

if w_1, w_2 : $T(v_1) = w_1$, $T(v_2) = w_2$

then $T(v_1 + v_2) = w_1 + w_2$

$\Rightarrow w_1 + w_2 \in \operatorname{Im}(T)$.

Example: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$

$T = \mu_A$

Find basis for the kernel and the image of T.

Kernel: Solve $AX = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 4 & 5 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} t$$

So $B_k = \left\{ \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \right\}$ is a basis for $\ker(T)$.

Image: The subspace of $\mathbb{R}^3$ formed by taking lin. comb. of the columns of the matrix.

Another name for the image – column space.

Any procedure for excluding "extra" vectors from the set of columns to form a lin. indep. set will work.

Try:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 4 & 5 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right]$$

reduces to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third column corr. to a free variable, so

$$r_1\begin{bmatrix}1\\2\\1\end{bmatrix} + r_2\begin{bmatrix}1\\4\\3\end{bmatrix} + r_3\begin{bmatrix}1\\5\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

has solutions with $r_3 \neq 0$.

By choosing $r_3 = -1$, $\begin{bmatrix}1\\5\\4\end{bmatrix}$ can be expressed as lin. comb. of the first two vectors, which must be lin. independent (no more free variables!)

So $B_I = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\4\\3\end{bmatrix} \right\}$ is a basis for the image of T.

<u>Example</u>:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$ Find bases for the kernel and the image.

<u>Kernel</u>:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Partial reduction:

$-R_1 + R_2 \to R_2$

$-R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{3} R_2 \to R_2$

$-6R_2 + R_3 \to R_3$

$-R_1 + R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-R_2 \to R_2$

$-2R_2 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t.$$

Basis of the kernel: $B_0 = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

Basis of the image:

$$B_I = \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$ (throw the last vector out: free variable)

or $B_I' = \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (a little simpler)

or $B_{II}'' = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (even simpler)

Exercises 6.6 : 20b, (3, 4)

one credit